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Riemann Surfaces and Dessin D'enfants

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Riemann surfaces and dessins d'enfants

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Abstract

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by Javier Alcaide Pérez

In this document, I have recovered what I studied during the years 2016 and 2017 with Professor Carlos A. A. Florentino. The first chapter covers the basic notions of the theory of Riemann Surfaces, some important results such as the Euler-Poincaré characteristic, or the Hurwitz formula, the definition of bundles and sheaves and a proof of the Riemann-Roch theorem. This theory is used in the second chapter to proof the main theorem of this thesis, Belyi's theorem:

"A compact Riemann surface S is defined over a number field if and only if there exists a Belyi map on S"

The proof requires us to introduce some results of "valuations" and "specializations".

In the third chapter, we give the first definitions and results of dessins d'enfants, and how they are related to the Riemann surfaces (and so, to the algebraic curves). We end the thesis giving some important examples and results of a more particular nature, related to the theory of dessins d'enfants, such as the Shabat polynomials.

Key Words: Riemann Surfaces, Dessin d'enfants, cohomology, Riemann.

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Resumo

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Neste documento recupera-se o que foi estudado ao longo dos anos de 2016 e 2017 com o professor Carlos A.A. Florentino.

Capítulo I

O primeiro capítulo aborda as noções básicas da teoria da Superfície de Riemann, alguns resultados importantes, como a "característica de Euler-Poincaré" ou a fórmula de Hurwitz, que relaciona o grau e a multiplicidade de uma aplicação entre duas superfícies X e Y com o gênero das duas superfícies X e Y da seguinte forma:

$$2g_X - 2 = \deg(F)(2g_Y - 2) + \sum_{p \in X} [m_p(F) - 1]$$

Também se introduze a definição de objetos como os "fibrados vetoriais", os "fibrados lineares", que não são mais que fibrados vetoriais unidimensionais, ou os "feixes de grupos". Ainda se da uma prova do teorema de Riemann-Roch, que usa a dualidade de Serre e define os grupos de cohomologia de Cech $H1(M, L)$ em função do espaço de seções holomórficas de um feixe linear:

$$H^p(M, E) \cong H^{1-p}(M, K \otimes \mathcal{O}(E^*))^*$$

Onde K e o fibrado vectorial canonico.

O teorema de Riemann-Roch fornece uma fórmula que relaciona as dimensões dos grupos de cohomologia ao gênero da superfície.

Theorem 0.1. *Seja E um feixe vetorial de rank m, em uma superfície de Riemann M, do gênero g. Então:*

$$\dim H^0(M, E) - \dim H^0(M, E^* \otimes K) = \deg(E) + m(1 - g)$$

Este teorema é provado usando a indução sobre o rango de E, m, e o fato de que de uma sucessão curta exata de feixes

$$0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{U} \to 0$$

sempre se pode encontrar uma sucessão comprida exata de grupos de cohomologia

$$\begin{aligned} 0 \to H^0(M, \mathcal{S}) \to H^0(M, \mathcal{T}) \to H^0(M, \mathcal{U}) \to \\ \to H^1(M, \mathcal{S}) \to H^1(M, \mathcal{T}) \to H^1(M, \mathcal{U}) \to ... \end{aligned}$$

Além disso, na última parte do primeiro capítulo, se demonstra a equivalência de três categorias:

- As superfícies de Riemann - As extensões finitas do corpo de funções racionais $\mathbb{C}(z)$, - A categoria de curvas algébricas projetivas.

A passagem da primeira para a segunda categoria é simples: basta considerar o corpo das funções meromórficas da superfície $M(S)$. Este corpo pode ser escrito em função de dois geradores como $\mathbb{C}(f, g)$ e, pelo teorema do elemento primitivo, pode-se encontrar um polinômio $F(X, Y)$ de tal forma que $F(f, g) = 0$. Assim, vamos de um objeto da segunda categoria para a terceira.

Capítulo II

Toda a teoria anterior é usada no segundo capítulo para demonstrar o principal resultado desta tese, o teorema de Belyi:

Theorem 0.2. *O facto de que uma superfície S de Riemann compacta esteja definida em um corpo numérico é equivalente a houver uma aplicação de Belyi f em S.*

Diz-se que uma superfície de Riemann S está definida em um corpo K se há uma equação polinomial F que a define de modo que F é definido no corpo K. Se chama aplicação de Belyi em S a uma aplicação que vai de S para a esfera de Riemann e que só tem pontos $0, 1$ e ∞ da esfera de Riemann como valores críticos.

A prova deste teorema requer a introdução de alguns resultados da teoria da "Evaluations" , uma generalizaçao do concepto de grau de um morfismo, e "Specializations", bem como a noção de homomorfismo de monodromia associado a uma ação.

Capítulo III

No terceiro capítulo, dão-se as primeiras definições e resultados do "Dessins d'enfants" uma estrutura que, aunque fue introduzida por Alexander Groethendieck quando estava a estudar o grupo absoluto de Galois, ja foi usada por Felix Klein para constuir una cobertura de 11 folhas da esfera de Riemann. E, tambén, cómo estas estruturas se relacionam com as superfícies de Riemann (e, portanto, com as curvas algébricas).

Um dessin d'enfants é uma superfície topológica X na qual um grafo bicolor é "desenhado". Esta estrutura combinatória permite dar-lhe à superfície X a estrutura de superfície de Riemann e, ainda, a estrutura de um revestimento ramificado da esfera de Riemann $\mathbb{P}^1$:

Dado um dessin d'enfants $\mathcal{D} = (S, G)$, sempre se pode encontrar uma aplicação Belyi f de S para a esfera de Riemann enviando os puntos brancos do grafo ao 0, os pontos pretos do grafo ao 1 e certos pontos selecionados dentro das faces de dessin d'enfants ao infinito.

Se demostra assim que existe uma correspondência um a um entre os Dessin d'enfants e os pares Belyi (S, F), onde S é uma superfície de Riemann e f é uma aplicação de Belyi em S.

Também se dão algumas ideias da monodromia dos dessin d'enfants e cómo um dessin de grau d também pode ser descrito como um subconjunto de F_2, o grupo livre de dois geradores que actua transitivamente em conjuntos de d elementos:

Dado que uma função Belyi, restringida à preimagem de $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, é um revestimento no sentido mais geral de topologia algébrica, existe uma relação fundamental entre os grupos fundamentais da superfície (excepto o grafo) e o grupo fundamental de $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, que é o grupo livre de dois elementos. A monodromia da dessin é o homomorfismo:

$$\rho : \pi(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to \Sigma_d$$

i.e.

$$\rho : F_2 \to \Sigma_d$$

que envia um nó em X baseado em $x0$ para a permutação que ele induz na fibra $f^{-1}(x0)$.

O grupo cartográfico de uma dessin é o subgrupo que e imagem do homomorfismo da monodromia:

$$\rho : F_2 \to \Sigma_d$$

Da-se um método para computar este grupo apenas com a combinatória do gráfico associado ao Dessin, e por qué este grupo é útil.

A tese continua a dar alguns exemplos importantes, e resultados de natureza mais específica, relacionados com a teoria da Dessins d'enfants, tais como polinômios Shabbat.

Da-se a definição e alguns resultados importantes dos dessins d´enfants regulares, que são os dessins mais simétricos que existem. Um exemplo de dessina regular é introducir um dos sólidos platônicos, como o tetraedro, na esfera de Riemann.

Finalmente, estuda-se a ação do grupo absoluto de Galois $Gal(\overline{\mathbb{Q}})$ sobre o conjunto de dessin d'enfants, e alguns de seus invariantes, como o número de arestas do grafo, o número de vértices brancos e pretos, o número de faces ou o grupo cartográfico. Além disso, da-se uma prova da "fidelidade" desta ação sobre o conjunto de dessins de gênero g.

A importância disto é que, dados dois elementos diferentes do grupo absoluto de Galois, estes induzem duas permutações diferentes de desins d'enfants de gênero g, isto é, estudando as permutações de dessins d'enfants pode-se obter informação relevante sobre o grupo $Gal(\overline{\mathbb{Q}})$.

A tese baseia-se, acima de tudo, em dois livros: "Curvas Algébricas e superfícies de Riemann", de Rick Miranda, e "Introdução às superfícies compactas de Riemann e Dessins d'enfants", de Ernesto Girondo e Gabino González-Diez, e aborda os primeiros conceitos das teorias de superfícies compactas de Riemann e dos dessin d'enfants de uma maneira bastante clara.

Peço desculpa pelo péssimo português que tive de usar, mas as regras da FCUL são rigorosas em relação às 1200 palavras que devem ser usadas no resumo desta língua.

Palavras chave: Riemann Surfaces, Dessin d'enfants, cohomology, Riemann.

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Contents

Para Lua...

Chapter 1

Introduction

1.1 Riemann Surfaces

Definition 1.1. A **Riemann Surface** S is a one-dimensional complex manifold, i.e. a two-real-dimensional smooth manifold with a maximal set of coordinate charts $(U_\alpha \subset S, \varphi_\alpha)$:

$$\varphi_\alpha : U_\alpha \to \mathbb{R}^2 \cong \mathbb{C}$$

such that the change of coordinates

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is an invertible holomorphic map for all α and all β.

Definition 1.2. Let S and S' be Riemann surfaces. A **holomorphic map**, or **morphism**, $f : S \to S'$ is a continuous map such that for every chart $\varphi_\alpha : U_\alpha \to \mathbb{C}$ on S and every chart $\phi_\beta : V_\beta \to \mathbb{C}$ on S', the function

$$\tilde{f} = \phi_\beta \circ f \circ \varphi_\alpha : \varphi_\alpha(U_\alpha) \to \phi_\beta(V_\beta)$$

is holomorphic.

A **meromorphic function** $f : S \to \mathbb{C}$ is a continuous map such that for every chart $\varphi_\alpha : U_\alpha \to \mathbb{C}$ on S , the function $\tilde{f}$ is meromorphic in the usual sense of complex analysis.

The set of meromorphic functions on a Riemann surface S, that is, the set of meromorphic maps $f : S \to \mathbb{C}$ is a field, and it is denoted by $\mathcal{M}(S)$.

Example 1.1. *Let $S = \mathbb{S}^2$ be the sphere $\{x^2+y^2+z^2 = 1\}$. The stereographic projection from the north pole $N = (0, 0, 1)$ and from the south pole $S = (0, 0, -1)$ give us two* *coordinate charts* $\phi_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ *and* $\phi_S : \mathbb{S}^2 \setminus \{S\} \to \mathbb{C}$. *A simple computation choosing the correct orientations on the tangent spaces shows us that* $\phi_S \circ \phi_N^{-1}(z) = \frac{1}{z}$, *which is an holomorphic function in* $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. *The sphere* $\mathbb{S}^2$ *with this complex structure is known as the Riemann Sphere, or the complex projective line* $\mathbb{P}^1$.

Consider now the rational function

$$f(z) = \frac{p(z)}{q(z)} = \frac{a_0 + a_1 z + ... + a_n z^n}{b_0 + b_1 z + ... + b_m z^m}$$

If we assume that the polynomials $p(z)$ *and* $q(z)$ *have no common zeroes, the function* f *defines a morphism from* $\mathbb{P}^1$ *to* $\mathbb{P}^1$. *In fact, every morphism* $f : \mathbb{P}^1 \to \mathbb{P}^1$ *is defined by a rational function.*

Proposition 1.3. *The field* $\mathcal{M}(\mathbb{P}^1)$ *is the field of rational functions in one variable* $\mathbb{C}(z)$.

Proof. Let f be a meromorphic function in $\mathbb{P}^1$, and suppose that $f(\infty) \neq \infty$. We can assume that by taking, if $f(\infty) = \infty$, the function $1/f$. Since $\mathbb{P}^1$ is compact, the number of poles of f is finite, say $a_1, ..., a_n$. For each pole a_i, we can write locally the Laurent series

$$f(z) = \sum_{k=1}^{r_i} \frac{\lambda_k^i}{(z - a_i)^k} + h_i$$

where h_i is holomorphic at a_i.

Therefore,

$$f - \sum_{i=1}^{n} \sum_{k=1}^{r_i} \frac{\lambda_k^i}{(z - a_i)^k}$$

is a holomorphic function in the Riemann sphere, which is compact, so by Liouville's Theorem, it must be constant, and thus, f must be a rational function. □

Definition 1.4. Let $f : S \to \mathbb{C}$ be a meromorphic function, and φ a chart around $p \in S$ such that $\varphi(p) = 0$[1]. Consider the Laurent expansion of $f \circ \varphi^{-1}$:

$$f \circ \varphi^{-1}(z) = a_n z^n + a_{n+1} z^{n+1} + ...$$

where $a_n \neq 0$.

The integer n is called the **order** of f at p and is denoted by $\mathrm{ord}_p(f)$.

[1]This kind of charts, where $\varphi(p) = 0$ are called charts centered at p.

Consider now two Riemann surfaces S_1 and S_2 and a morphism between them

$$f : S_1 \to S_2$$

Choose two points $p \in S_1$ and $q = f(p) \in S_2$, and a chart ψ centered at q. The positive integer

$$m_p(f) := \mathrm{ord}_p(\psi \circ f)$$

is called the **multiplicity** of f at p.

If $m_p(f) \geq 2$, we say that p is a **branch point**, or a ramification point, with branching order $m_p(f)$, and we will say that $q = f(p)$ is a **branch value**.

Definition 1.5. Let $F : S_1 \to S_2$. The integer

$$\deg(F) = \sum_{F(p)=q} m_p(F)$$

is not dependant of the point q and it is called the degree of F.

1.1.1 The Euler-Poincaré characteristic and the Hurwitz's formula

It is well known that the compact oriented topological surfaces can be classified by their genus g.

Theorem 1.6. *Let S be topological surface such that it is connected, compact and orientable. Then, it is homeomorphic to a sphere ($g = 0$), or to a connected sum of g topological tori. The integer number g is called the genus of the surface.*

Proof. See Muñoz [1] or section 2.4.A in Jost [2]. □

Remark 1.7. Every Riemann surface is orientable, as every transition map of an holomorphic atlas is holomorphic and thus, it preserves the orientation.

The proof of the above classification theorem uses the notion of triangulation:

Definition 1.8. A triangulation of a surface S is a finite set of triangles [2] $\{T_k\}$ such that:

1. The surface S is the union of the triangles, $S = \bigcup T_k$.

2. The intersection of two triangles is either empty, one vertex, or one edge.

[2] The definition of triangulation can be given with polygons with an arbitrary number of sides, and they will be transformed into triangles by the first refining operation.

Suppose we have a triangulation on a surface S. We can *refine* the triangulation with the following operations:

1. We can add a vertex in the interior of a triangle T, and three edges from this new vertex to the vertices of the triangle T. With this operation, we add 2 triangles, 3 edges and one vertex.
2. Take two triangles T_1 and T_2 such that they share an edge. Choose a point v in the interior of the edge and add two edges to each of the opposite vertices of T_1 and T_2.

Remark 1.9. Any two triangulations on a surface S have a common refinement.

Theorem 1.10. *Every compact Riemann surface can be triangulated.*

Sketch of the proof. The idea of the proof is to use the existance of a Riemannian metric, and thus of geodesics, on Riemann surfaces.

One can choose a couple of points such that each of them has a certain number of other ones near enough to join them by geodesics in a way that this geodesics are unique. Those geodesics divide the surface in small polygons which we can them subdivide in small triangles using the first operation. This geodesics may intersect each other, but this problem can be solved by noticing that in this intersections, the geodesics intersect with a nonzero angle.

Details can be seen in section 2.3.A of Jost [2]. ☐

Another way to define the genus of a surface is the following:

Proposition 1.11. *Let S be a compact orientable surface of genus g. Let v, e and f be the number of vertices, edges and faces of a given triangulation. Then the Euler-Poincaré characteristic*

$$\chi(S) := v - e + f$$

does not depend of the triangulation and

$$\chi(S) = 2 - 2g$$

Proof. First, note that the two operations we gave for building refinements of triangulations preserve the Euler characteristic: both of them add one vertex, 2 faces and 3 edges, so the Euler characteristic is constant under refinement. Since every two triangulations have a common refinement they must have the same Euler characteristic.

Now, it is easy to compute the Euler number for the sphere (2), the Euler number of a cylinder is 0, and the Euler characteristic of a closed disk is 1.

One can increase the genus of a surface removing two disks and attaching a cylinder along the two bounding circles. This operation, of course, decrease the Euler number by 2, one for each of the circles erased, so the Euler characteristic decreases by two if the genus increases by one.

Now, by induction over the genus of the surface:

If the genus is 0, $S = \mathbb{S}^2$ and the Euler characteristic is 2.

Suppose that for genus g, the Euler characteristic is $2 - 2g$, then, for surfaces of genus $g + 1$, the Euler characteristic will be $2 - 2g - 2 = 2 - 2(g + 1)$. $\square$

Using the Euler characteristic, we can prove the following result, which relates the degree and ramification of a map $F : X \to Y$ with the genera of the surfaces X and Y.

Theorem 1.12 (Hurwitz's Formula)**.** *Let X, Y be two compact Riemann surfaces of genera g_X and g_Y and $F : X \to Y$ be a nonconstant holomorphic map between them. Then*

$$2g_X - 2 = \deg(F)(2g_Y - 2) + \sum_{p \in X} [m_p(F) - 1]$$

Proof. First, since X is compact, the number of ramification points of F must be finite, and so, the sum in the formula is a finite sum. Let $\{q_1, ..., q_n\}$ be the ramification values of F, and consider a triangulation on Y such that this points are vertices of the triangles. (This can be done by taking a triangulation and adding, with the two refiniment operations, vertices in the branching values). Let v, e and f the number of vertices, edges and faces of this triangulation. We can lift this triangulation to a triangulation in X with v' vertices, e' edges and f' faces, taking the preimage by F.

Since there are no ramification points on the interior of the triangles, each triangle in Y lifts to $\deg(F)$ triangles in X, and thus, $f' = \deg(F)f$. The same argument shows that $e' = \deg(F)e$.

Now, choose a vertex $q \in Y$.

$$\begin{aligned} |F^{-1}(q)| &= \sum_{p \in F^{-1}(q)} 1 \\ &= \deg(F) + \sum_{p \in F^{-1}(q)} [1 - m_p(F)] \end{aligned}$$

and thus, the number of preimages of vertices of Y, v' is

$$\begin{aligned} v' &= \sum_{vertices\, q} [\deg(F) + \sum_{p\in F^{-1}(q)} (1 - m_p(F))] \\ &= \deg(F)v - \sum_{vertices\, q} \sum_{p\in F^{-1}(q)} (m_p(F) - 1) \\ &= \deg(F)v - \sum_{p\in X} (m_p(F) - 1) \end{aligned}$$

Therefore

$$\begin{aligned} 2g_X - 2 &= -\chi(X) = -v' + e' - f' \\ &= -\deg(F)v + \sum_{p\in X}(m_p(F) - 1) + \deg(F)e - \deg(F)f \\ &= \deg(F)(-v + e - f) + \sum_{p\in X}(m_p(F) - 1) \\ &= \deg(F)(2g_Y - 2) + \sum_{p\in X}[\mathrm{mult}_p(F) - 1]. \end{aligned}$$

□

1.2 Vector Bundles and Sheaves

In this section, we will present the proof of the Riemann-Roch theorem, which is important to show the existance of meromorphic functions in compact Riemann surfaces. With this purpose, we need to introduce some notions in the theory of Vector bundles and sheaves, and some results of the theory of cohomology groups.

Definition 1.13. Let M be a Riemann surface. A holomorphic line bundle L over M is a two-dimensional complex manifold together with a holomorphic projection $\pi : L \to M$ such that

(1) For each $p \in M$, the fiber $\pi^{-1}(p)$ has the structure of a complex one-dimensional vector space.

(2) Each point $p \in M$ has a neighbourhood U and a homeomorphism φ_U such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, phantom, "\overset{\varphi_U}{\cong}"] \arrow[dr, "\pi"] & & U \times \mathbb{C} \arrow[dl] \\ & U & \end{tikzcd}$$

is commutative.

(3) The map $\varphi_V \circ \varphi_U^{-1}$ has the form

$$\begin{aligned}\varphi_V \circ \varphi_U^{-1} : (U \cap V) \times \mathbb{C} &\to (U \cap V) \times \mathbb{C} \\ (p, \omega) &\mapsto f(p)\omega\end{aligned}$$

in the intersection, where f is a non-vanishing holomorphic function.

The function f above, is known as the transition function of the line bundle an it is usually denoted by g_{UV}.

The homeomorphism φ_U is known as a local trivialization over U.

Definition 1.14. A holomorphic section of a line bundle L over a Riemann surface M is a holomorphic map $s : M \to L$ such that $\pi \circ s = Id_M$.

Example 1.2. *1. The line bundle $M \times \mathbb{C}$ is the trivial bundle over M.*

2. The canonical bundle K is the bundle of holomorphic 1-forms. Given two local coordinates z and $\tilde{z}$, it is easy to see that the transition functions are $\frac{d\tilde{z}}{dz}$.

3. Choose a point $p \in M$ and consider the open sets U_0, an open neighbourhood of p with local coordinate z centered at p, and $U_1 = M \setminus \{p\}$. We can set z to be the transition function of a line bundle, by gluing together $U_0 \times \mathbb{C}$ and $U_1 \times \mathbb{C}$ using

$$\varphi(p, \omega) = (p, z(p)\omega)$$

We denote this line bundle by L_p. The line bundle L_p has a canonical section s_p given by the functions z in U_0 and 1 in U_1. This section is well-defined since, in $U_0 \cap U_1$, $z = z.1 = g_{01}.1 = g_{U_0U_1}.1$. This section s_p has a single zero at p and only there.

Definition 1.15. Let X be a topological space. A presheaf of groups $\mathcal{S}$ on X is a collection of groups $\mathcal{S}(U)$, one for each open set U of X, and a collection of group homomorphisms $\rho_V^U : \mathcal{S}(U) \to \mathcal{S}(V)$ whenever $V \subseteq U$, such that:

1. The group $\mathcal{S}(\emptyset)$ is the trivial group.

2. The homomorphism ρ^U_U is the identity on $\mathcal{S}(U)$.

3. If $W \subseteq V \subseteq U$, then $\rho^U_W = \rho^V_W \circ \rho^U_V$.

The homomorphisms ρ^U_V are called the restriction maps. The elements of $\mathcal{S}(U)$ are called sections over U, and the elements of $\mathcal{S}(X)$ are global sections of $\mathcal{S}$.

Definition 1.16. A sheaf is a presheaf that satisfies the following axioms:

1. **Identity axiom**: If $\{U_i\}_{i\in I}$ is an open cover of U, and f_1, f_2 are sections of U such that $\rho^U_{U_i}(f_1) = \rho^U_{U_i}(f_2)$ for all i, then $f_1 = f_2$.

2. **Gluing axiom**: If $\{U_i\}_{i\in I}$ is an open cover of U, then, given $f_i \in \mathcal{S}(U_i)$ such that $\rho^{U_i}_{U_i\cap U_j}(f_i) = \rho^{U_j}_{U_i\cap U_j}(f_j)$, then there exist a section $f \in \mathcal{S}(U)$ such that $\rho^U_{U_i}(f) = f_i$ for all $i \in I$.

Example 1.3. *Some examples of sheaves are:*

1. *Given a Riemann surface X, $\mathcal{S}(U) = \mathcal{O}(U) = \{f : U \to \mathbb{C}, f \text{ is holomorphic in } U\}$.*

2. *Given a Riemann surface X and a line bundle L over X, $\mathcal{S}(U) = \mathcal{O}(L)(U) = \{s : s \text{ holomorphic section of } L \text{ over } U\}$*

3. *Let X be a topological space, $p \in X$ and G be a group. We can define the Skyscraper sheaf on p by*
$$\mathcal{S}(U) = \begin{cases} G & \text{if } p \in U \\ \{e\} & \text{if } p \notin U \end{cases} \tag{1.1}$$

Definition 1.17. The stalk $\mathcal{S}_p$ of a sheaf $\mathcal{S}$ at the point p is the set of germs of $\mathcal{S}$ at p

$$\{(f, U) : U \text{ open}, p \in U, f \in \mathcal{S}(U)\}$$

modulo the equivalence relation given by:

$(f, U) \sim (g, V)$ if there is some open set $W \subseteq U \cap V$, with $p \in W$ and $\rho^U_W(f) = \rho^V_W(g)$. Equivently, it is the colimit of all groups $\mathcal{S}(U)$ over all open neighbourhoods U of p.

Now, we would like to define the Čech cohomology groups $H^p(M, \mathcal{S})$ of M with coefficients in $\mathcal{S}$. Let $\{U_\alpha\}_{\alpha\in A}$ be a locally finite open covering of M. We denote by

$U_{i_0,\ldots,i_p} = \bigcap_{i=0,\ldots,p} U_i$.
Consider the groups

$$C^p(U, \mathcal{S}) = \Pi_{(i_0,\ldots,i_p)\in A^{p+1}} \mathcal{S}(U_{i_0,\ldots,i_p})$$

Consider a section $s \in C^p(U, \mathcal{S})$, we denote by $s_{i_0,\ldots,i_p}$ the value of s on $U_{i_0,\ldots,i_p}$.

We define now the coboundary operator

$$\delta : C^p(U, \mathcal{S}) \to C^{p+1}(U, \mathcal{S})$$

by

$$(\delta(s))_{i_0,\ldots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0,\ldots,\hat{i_j},\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}}$$

[3]

Proposition 1.18. *The coboundary map δ satisfies*

$$\delta \circ \delta = 0$$

and, so, the chain

$$\ldots \to C^p(U, \mathcal{S}) \xrightarrow{\delta} C^{p+1}(U, \mathcal{S}) \to \ldots$$

is a chain complex, that is, the image of each homomorphism, is included in the kernel of the next.

Proof. For every $p-1$-coboundary $g = \delta(f)$, we have that for every $i_0, \ldots, i_{p+1}$:

$$\begin{aligned}
\delta(g)_{i_0,\ldots,i_{p+1}} = \delta(\delta(f))_{i_0,\ldots,i_{p+1}} &= \sum_{j=0}^{p+1} (-1)^j f_{i_0,\ldots,\hat{i_j},\ldots,i_{p+1}} \\
&= \sum_{j=0}^{p+1} (-1)^j (\sum_{l=0}^{j-1} (-1)^l f_{i_0,\ldots,\hat{i_l},\ldots\hat{i_j},\ldots,i_{p+1}} + \sum_{l=j+1}^{p+1} (-1)^{l-1} f_{i_0,\ldots,\hat{i_j},\ldots\hat{i_l},\ldots,i_{p+1}}) \\
&= \sum_{j>l\geq 0}^{p+1} ((-1)^{j+l} f_{i_0,\ldots,\hat{i_l},\ldots\hat{i_j},\ldots,i_{p+1}}) + \sum_{l>j\geq 0}^{p+1} ((-1)^{j+l-1} f_{i_0,\ldots,\hat{i_j},\ldots\hat{i_l},\ldots,i_{p+1}}) = 0
\end{aligned}$$

□

Definition 1.19. The p-th cohomology group of $\mathcal{S}$ relative to the covering $\{U_\alpha\}_{\alpha\in A}$ is

$$H^p(M, \mathcal{S}) := \frac{ker(\delta : C^p \to C^{p+1})}{im(\delta : C^{p-1} \to C^p)}$$

[3] where $s|_U$ means $\rho^V_U(S)$ if s is a section of $\mathcal{S}$ on V and $U \in V$.

Remark 1.20. In order to define cohomology groups on a surface S which don't depend on the covering, we use the refinement of coverings to define direct limit of the cohomology groups. To check the details of this construction, see Miranda [3].

Example 1.4. *Let L be a holomorphic line bundle over a Riemann surface M, and $\mathcal{O}(L)$ the sheaf of holomorphic sections of L. If $f \in C^0$, then $\delta(f) = f_\beta - f_\alpha$ and so, if $\delta(f) = 0$ we can glue together the local sections f_α to give a global section, i.e.*

$$H^0(M, L) := H^0(M, \mathcal{O}(L)) = ker(\delta)$$

is the space of global holomorphic sections of the line bundle L.

Proposition 1.21. *Let M be a compact Riemann surface, and let L be a line bundle over it. Then, $H^0(M, L)$ is a finite dimensional vector space. Moreover, if M has genus g and K is the canonical line bundle (see example 1.2),*

$$\dim(H^0(M, K)) = g$$

Proof. We have seen in the last example that the space $H^0(M, L)$ is the space of holomorphic sections of the line bundle L. We can give this space the structure of a vector space with the following operations:

1. We can add sections pointwise

$$(s + t)(p) := s(p) + t(p)$$

2. We can multiply sections by scalars

$$(\lambda s)(p) := \lambda(s(p))$$

The proof of the finiteness of the dimension of this vector space and its calculation can be seen in Gunning [4]. $\square$

Example 1.5. *Let M be a Riemann surface. Then:*

1. *The group $H^p(M, L)$ is trivial if $p > 1$.*

2. *The group $H^p(M, \underline{\mathbb{C}})$ is trivial if $p > 2$.*

3. The group $H^p(M, \underline{\mathbb{Z}})$[4] is trivial if $p > 2$.

Remark 1.22. The groups $H^k(M, \underline{\mathbb{C}})$ are identified with the **de Rham** cohomology groups $H^k_{dR}(M)$, and the groups $H^k(M, \underline{\mathbb{Z}})$ with the singular cohomology groups. This can be seen in chapter IX of Miranda [3].

The definition of cohomology groups may result a little confusing but if we work only in Riemann surfaces, as we have seen in the above last example, the only groups we will care are H^0, H^1 and, sometimes, H^2. This allows us to characterice cohomology groups in a very explicit way. Namely, we have the following theorem, Serre's duality, which defines the group $H^1(M, L)$ in terms of the space of holomorphic sections of a line bundle.

Theorem 1.23 (Serre's duality)**.** *Let L be a line bundle over a compact Riemann surface M. Let K be the canonical bundle we defined in example 1.2. Then,*

$$H^1(M, L) \cong H^0(M, K \otimes L^*)^*$$

Proof. See Chapter IV in Miranda [3]. □

Consider now a short exact sequence of sheaves on a surface M

$$0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{U} \to 0$$

Definition 1.24. We say that a short sequence of sheaves is exact if, for every $p \in M$, there exists a neighbourhood V of p such that, for every open set $U \subseteq V$ such that $p \in V$, we have that the short sequence

$$0 \to \mathcal{S}(U) \to \mathcal{T}(U) \to \mathcal{U}(U) \to 0$$

is exact.

Given a short exact sequence of sheaves

$$0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{U} \to 0$$

we can construct a long exact sequence of cohomology groups

$$\begin{aligned} 0 \to H^0(M, \mathcal{S}) \to H^0(M, \mathcal{T}) \to H^0(M, \mathcal{U}) \to \\ \to H^1(M, \mathcal{S}) \to H^1(M, \mathcal{T}) \to H^1(M, \mathcal{U}) \to \dots \end{aligned}$$

[4]The sheaves $\underline{\mathbb{C}}$ and $\underline{\mathbb{Z}}$ are the constant sheaves, that is, $\mathcal{S}_p$ is respectively $\mathbb{C}$ and $\mathbb{Z}$ for every p

Example 1.6. *Let L be a line bundle on M, and L_p the line bundle associated to $p \in M$ defined in example 1.2. We can define a short exact sequence of sheaves*

$$0 \to \mathcal{O}(LL_p^{-1}) \xrightarrow{s_p} \mathcal{O}(L) \to \mathcal{O}_p(L) \to 0$$

where the stalk $\mathcal{O}_p(L)$ can be interpreted as the sheaf of sections of L over $U \cap \{p\}$, whose space of global sections is just $H^0(M, L_p) = \pi^{-1}(p) \cong \mathbb{C}$. This short exact sequence gives rise to a long exact sequence of cohomology groups

$$0 \to H^0(M, LL_p^{-1}) \to H^0(M, L) \xrightarrow{\varphi} \mathbb{C} \xrightarrow{\delta} H^1(M, LL_p^{-1}) \to \dots$$

If δ is non-zero, φ must vanish, and thus,

$$H^0(M, LL_p^{-1}) \cong H^0(M, L)$$

given by multiplication by s_p and, as s_p vanishes at p, f δ is not zero, all global sections of L must vanish at p.

Now, we would like to study the space $H^1(M, \mathcal{O}^*)$, also known as the Picard group: the space of equivalence classes of line bundles.

Consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{e^{2\pi i -}} \mathcal{O}^* \to 1$$

It gives rise to a long exact sequence of cohomology groups

$$\begin{aligned} 0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to H^1(M, \mathbb{Z}) \to \\ \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^*) \to \\ \to H^2(M, \mathbb{Z}) \to H^2(M, \mathcal{O}) \to \dots \end{aligned}$$

Since the exponentiation is surjective onto $\mathbb{C}^*$, the map $H^1(M, \mathbb{Z}) \to H^1(M, \mathcal{O})$ is injective, and, since $H^2(M, \mathcal{O}) = 0$ (see example 1.5), we have the sequence

$$0 \to \frac{H^1(M, \mathcal{O})}{H^1(M, \mathbb{Z})} \to H^1(M, \mathcal{O}^*) \xrightarrow{\delta} H^2(M, \mathbb{Z}) \to 0$$

As M is a compact 2-real manifold, $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$, and so, it makes sense to define de degree of a line bundle by the following:

Definition 1.25. The degree of a line bundle L is $\deg(L) := \delta([L]) \in \mathbb{Z}$.

Remark 1.26. 1. Since δ is a group homomorphism, $\deg(L_1 \otimes_2) = \deg(L_1) + \deg(L_2)$.

2. A simple application of the Mayer-Vietoris sequence, shows that for every $p \in M$, $\deg(L_p) = 1$.

Proposition 1.27. *Let L be a line bundle over a Riemann surface M. If a section $s \in H^0(M, L)$ vanishes at the points $p_1, ..., p_n$ with multiplicities $m_1, ..., m_n$, then the degree of L is*

$$\deg(L) = \sum_{i=0}^{n} m_i$$

Proof. Consider the non-vanishing section of $LL_{p_1}^{-m_1}...L_{p_n}^{-m_n}$ given by $ss_{p_1}^{-m_1}...s_{p_n}^{-m_n}$ (see Example 1.2). Therefore, $LL_{p_1}^{-m_1}...L_{p_n}^{-m_n}$ is a trivial bundle and its degree is

$$\deg(LL_{p_1}^{-m_1}...L_{p_n}^{-m_n}) = \deg(L) - (m_1 + ... + m_n) = 0$$

and so, by remark 1.26 $\deg(L) = \sum_{i=0}^{n} m_i$. $\square$

Now, we introduce the concept of vector bundle, which generalices the notion of line bundle.

Definition 1.28. Let M be a Riemann surface. A holomorphic vector bundle of rank m, E is a complex manifold together with a holomorphic projection $\pi : E \to M$ such that:

(1) For each point $p \in M$, the fiber $\pi^{-1}(p)$ has the structure of m-dimensional vector space.

(2) For each point $p \in M$, there exists a neighbourhood U and a homeomorphism φ_U such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi_U", "\cong"'] \arrow[dr, "\pi"'] & & U \times \mathbb{C}^m \arrow[dl] \\ & U & \end{tikzcd}$$

is commmutative.

(3) The map $\varphi_V \circ \varphi_U^{-1}$ has the form

$$(p, \omega) \mapsto (p, g_{UV}(p)\omega)$$

where $g_{UV} : U \cap V \to GL(m, \mathbb{C})$ is a holomorphic map to the space of complex invertible matrices. These maps are known as transition functions.

Remark 1.29. Given a vector bundle (E, π) over a Riemann surface M, we can construct a line bundle over M, by taking the highest exterior power

$$\det(E) := \bigwedge^{m} E$$

This line bundle is called the determinant of E, and its transition functions are given by the determinants of the transition functions of E, $\det(g_{UV})$.

Definition 1.30. The degree of a vector bundle E is the degree of its determinant

$$\deg(E) := \deg(\deg(E)).$$

In the same way we did with line bundles, we can compute the p-th cohomology group of the sheaf of holomorphic sections of a vector bundle $\mathcal{O}(E)$. These groups result to be trivial if $p > 1$.

$$H^p(M, E) := H^p(M, \mathcal{O}(E)) = 0 \text{ if } p > 1$$

Moreover, Serre's duality 1.23 also holds for vector bundles:

Theorem 1.31. *Let E be a vector bundle of rank m over a compact Riemann surface M. Let K be the canonical bundle we defined in example 1.2. Then,*

$$H^p(M, E) \cong H^{1-p}(M, K \otimes \mathcal{O}(E^*))^*$$

Proof. The proof can be seen in, for example, the book of Gunning [4]. □

1.2.1 The Riemann-Roch Theorem

We now prove the main theorem of this section: The Riemann-Roch theorem, which relates the dimensions of the cohomology groups $H^p(M, E)$.

Theorem 1.32 (Riemann-Roch)**.** *Let E be a vector bundle of rank m over a compact Riemann surface M, of genus g. Then,*

$$\dim H^0(M, E) - \dim H^0(M, E^* \otimes K) = \deg(E) + m(1 - g)$$

Proof. We will prove the result by induction on the rank m.
First, suppose E is a line bundle L. Then, if L is the trivial bundle, the sheaf of

sections $\mathcal{O}(L)$ is nothing more but the sheaf of holomorphic functions $\mathcal{O}$, and since M is compact, $\mathcal{O}(L) = \mathcal{O} \cong \mathbb{C}$. Using Serre's duality 1.31, we have that $\dim H^1(M, \mathcal{O}) = \dim H^0(M, K \times \mathcal{O}) = g$. Therefore,

$$\dim H^0(M, \mathcal{O}) - \dim H^1(M, \mathcal{O}) = 1 - g$$

On the other hand,

$$\deg(\mathcal{O}) + \mathrm{rk}(\mathcal{O})(1 - g) = 0 + 1(1 - g) = 1 - g$$

and, so,

$$\dim H^0(M, \mathcal{O}) - \dim H^1(M, \mathcal{O}) = \deg(\mathcal{O}) + \mathrm{rk}(\mathcal{O})(1 - g)$$

So the result is true for the trivial bundle. Now we claim that if the result holds for a line bundle L, then it also holds for the line bundles LL_p and LL_p^{-1}. Consider the short exact sequence

$$0 \to \mathcal{O}(L) \xrightarrow{s_p} \mathcal{O}(LL_p) \to \mathcal{O}_p(LL_p) \to 0$$

which gives rise to the long exact sequence

$$0 \to H^0(M, L) \to H^0(M, LL_p) \to \mathbb{C} \to H^1(M, L) \to H^1(M, LL_p) \to 0$$

Since the sequence is exact, the alternating sum of the dimensions of the spaces must vanish, and so,

$$\dim H^0(M, L) - \dim H^0(M, LL_p) + 1 - \dim H^1(M, L) + \dim H^1(M, LL_p) = 0$$

and then,

$$\dim H^0(M, L) - \dim H^1(M, L) + 1 = \dim H^0(M, LL_p) - \dim H^1(M, LL_p)$$

As we are supposing the formula to hold for L, we have that

$$\dim H^0(M, LL_p) - \dim H^1(M, LL_p) = \deg(L) + 1 + \mathrm{rk}(L)(1 - g) = \deg(LL_p) + (1 - g)$$

And so, the formula holds for the line bundle LL_p. A similar argument shows that the result also holds for LL_p^{-1}.

Now we would like to see that every line bundle can be written as product of L_p's and L_q^{-1}'s so, since the result is true for the line bundle, it would be true, inductively, for every line bundle. With this purpose, consider the sheaf $\mathcal{S}$ defined locally as the quotient space of $f(z) \mapsto z^n f(z)$ where z is a coordinate vanishing at p, and the following short exact sequence:

$$0 \to \mathcal{O}(L) \xrightarrow{s_p^n} \mathcal{O}(LL_p^n) \to \mathcal{S} \to 0$$

which induces

$$0 \to H^0(M, L) \to H^0(M, LL_p^n) \to \mathbb{C}^n \to H^1(M, L) \to H^1(M, LL_p^n) \to 0$$

Again, the alternating sum of the dimensions must be 0 and so,

$$\begin{aligned} \dim H^0(M, LL_p^n) &= n + \dim H^1(M, LL_p^n) + \dim H^0(M, L) - \dim H^1(M, L) \\ &\geq n + \dim H^0(M, L) - \dim H^1(M, L) \end{aligned}$$

If we take a large enough n, the dimension of $H^0(M, LL_p^n)$ is positive, so we can find a holomorphic section $s \in \mathcal{O}(LL_p^n)$. Suppose ths section s vanishes at the points $p_1, ..., p_k$ with multiplicities $m_1, ..., m_k$. Then the section $ss_{p_1}^{-m_1}...s_{p_k}^{-m_k}$ is a non-vanishing section trivializing the line bundle $LL_p^n L_{p_1}^{-m_1}...L_{p_k}^{-m_k}$, and so,

$$L \cong L_{p_1}^{m_1}...L_{p_k}^{m_k} L_p^{-n}$$

This proves the result for bundles of rank 1.

Suppose now that E is a vector bundle of rank m over M, and suppose inductively that the formula is true for all vector bundles of lower rank.

We would like to find a line bundle L as a subbundle of E, that is, to find a section of $\mathrm{Hom}(L, E) = L^* \otimes E$. Consider the short exact sequence

$$0 \to \mathcal{O}(L) \xrightarrow{s_p^n} \mathcal{O}(E \otimes L_p^n) \to \mathcal{S} \to 0$$

Just as above, if we take n large enough, the dimension $\dim H^0(E \otimes L_p^n)$ must be positive, so we can find a section $s \in \mathcal{O}(E \otimes L_p^n)$. If s vanishes in $p_1, ..., p_k$ with multiplicities $m_1, ..., m_k$, then the section $ss_{p_1}^{-m_1}...s_{p_k}^{-m_k}$ is a non-vanishing section of $E \otimes L^*$, where $L^* = L_p^n L_{p_1}^{-m_1}...L_{p_k}^{-m_k}$. Thus, we have found an inclusion $L \subset E$ as vector bundles.

Now we can consider the quotient $m-1$ vector bundle Q and the short exact sequence of sheaves

$$0 \to \mathcal{O}(L) \to \mathcal{O}(E) \to \mathcal{O}(Q) \to 0$$

and the induced long exact sequence of cohomologies:

$$0 \to H^0(M, L) \to H^0(M, E) \to H^0(M, Q) \to H^1(M, L) \to H^1(M, E) \to H^1(M, Q) \to 0$$

and, again, the alternating sum of the dimensions must vanish, so, by the induction hypothesis,

$$\dim H^0(M,E) - \dim H^1(M,E)$$
$$= \dim H^0(M,L) - \dim H^1(M,L) + \dim H^0(M,Q) - \dim H^1(M,Q)$$
$$= \deg(L) + 1(1-g) + \deg(Q) + (m-1)(1-g)$$
$$= \deg(L \otimes Q) + m(1-g) = \deg(E) + m(1-g)$$

which ends the proof. □

1.3 Equivalence of categories

In this section, our aim is to give a equivalence of categories between:

1) The category of Riemann surfaces.
2) The category of finite extensions of $\mathbb{C}(z)$.
3) The category of smooth projective algebraic curves.

Remark 1.33. In fact, we won't give a equivalence between the first two categories, but an equivalence between the first category and the opposite category of the second one, that is, we will define a contravariant functor from (1) two (2) that is an equivalence of categories.

Before providing these equivalences, we need to see some results about the compactification of Riemann surfaces, and the extension of meromorphic maps.

Lemma 1.34. *Let Y be a compact Riemann surface, $\Sigma \subset Y$ a finite subset, and consider $Y^* = Y \setminus \Sigma$. Let $f* : X^* \to Y^*$ be a unramified covering of degree n. Then, there exists a unique compact Riemann surface S, such that:*

1. *The Riemann surface S contains X^*, and $S \setminus X^*$ is a finite set.*
2. *The map $f*$ extends to a unique morphism $f : S \to Y$.*

Proof. See section 1.2.7 in Girondo and González-Diez [5]. □

Proposition 1.35. *Let S_1, S_2 be compact Riemann surfaces, and $\Sigma_1 \subset S_1$ and $\Sigma_2 \subset S_2$ be finite subsets. Assume that S_1^* and S_2^* are isomorphic. Then S_1 and S_2 are isomorphic too.*

Proof. Let $\varphi : S_1^* \to S_2^*$ be an isomorphism. As S_2 is hausdorff, we can find disjoiunt coordinate discs $V_1, ..., V_n$, one around each point of Σ_2.
Consider $x \in \Sigma_1$, and let $U \subset S_1$ be a coordinate disc around it such that $U \cap S_1^* = U \setminus \{x\}$. Consider a sequence of points $\{x_n\} \subset U \cap S_1^*$ converging to x, and let $y \in S_2$ be a limit point of $\{\varphi(x_n)\}$ in S_2. Then, $y \in \Sigma_2$ since in other case, the limit point of the sequence $\{x_n\}$ would be $x = \varphi^{-1}(y) \in S_1^*$, which is a contradiction. Therefore, if U is small enough, $\varphi(U \setminus \{x\}$ is contained in $V_1 \cup ... \cup V_n$, and, since they are disjunt and S_2^* is connected, $\varphi(U \setminus \{x\}$ must be in one of them, say V_i. Now, since this is a removable singularity, we can extend the map $\varphi : U \setminus \{x\} \to S_2$ to the whole U, setting $\varphi(x) = y$. Proceeding in this way with all the points, we would get a holomorphic map

$$\hat{\varphi} : S_1 \to S_2$$

whith degree 1, that is, an isomorphism. □

Now, consider a irreducible polynomial $F(X, Y) \in \mathbb{C}[X, Y]$, and let F_X and F_Y be their derivatives with respect to X and Y.

Theorem 1.36. *Let*

$$\begin{aligned} F(X,Y) &= p_0(X)Y^n + p_1(X)Y^{n-1} + ... + p_n(X) \\ &= q_0(Y)X^m + q_1(Y)X^{m-1} + ... + q_m(Y) \end{aligned}$$

be an irreducible polynomial in $K[X, Y]$ with $n, m \geq 1$.
Define

$$S_F^X = \{(x,y) \in \mathbb{C}^2 : F(x,y) = 0, F_Y(x,y) \neq 0, p_0(x) \neq 0\}$$

and

$$S_F^Y = \{(x,y) \in \mathbb{C}^2 : F(x,y) = 0, F_X(x,y) \neq 0, q_0(y) \neq 0\}$$

Then:

(i) S_F^X and S_F^Y are connected Riemann surfaces, on which the coordinate functions $\boldsymbol{x}$ and $\boldsymbol{y}$ are holomorphic functions respectively.

(ii) There exists a unique compact and connected Riemann surface $S = S_F$ that contains S_F^X and S_F^Y.

(iii) The coordinate functions $\mathbf{x}$ and $\mathbf{y}$ extend to meromorphic functions on S.

(iv) The branching points of $\boldsymbol{x}$ (resp. $\boldsymbol{y}$) lie in the finite set $S \setminus S_F^X$ (resp. $S \setminus S_F^Y$.

Proof. The holomorphic structure of S_F^X can be defined by using the implicit function theorem to *solve* $\mathbf{y}$ in terms of $\mathbf{x}$.
With this structure, it is easy to see that the coordinate functions are holomorphic, and that the coordinate function $\mathbf{x} : S_F^X \to \mathbf{x}(S_F^X) \subset \hat{\mathbb{C}}$ is a covering map of degree n. As the polynomials F and F_Y have only finitely many common zeros, $\mathbf{x}(S_F^X)$ must cover the whole $\hat{\mathbb{C}}$ except finitely many values $\{a_1, ..., a_r, \infty\}$.

Consider a connected component W of S_F^X. The restriction of $\mathbf{x}$ to W

$$\mathbf{x} : W \longrightarrow \hat{\mathbb{C}} \setminus \{a_1, ..., a_r, \infty\}$$

is still a covering map of degree $d \leq n$, so, by lemma 1.34 we can find a compact Riemann surface $\hat{W}$ and a unique morphism $\mathbf{x} : \hat{W} \longrightarrow \hat{\mathbb{C}}$ extending $\mathbf{x}$. To see that S_F^X is connected, we want to prove that $W = S_F^X$. Consider the symmetric functions

$$s_1(x) = \sum y_i(x), s_2(x) = \sum_{i<j} y_i(x)y_j(x), ..., s_d(x) = \prod_{i=1}^{d} y_i(x)$$

where $(x, y_1(x))$,...,$(x, y_d(x))$ are the preimages of $x \in \hat{\mathbb{C}} \setminus \{a_1, ..., a_r, \infty\}$ via the covering map $\mathbf{x}$. The functions $s_i(x)$ are well-defined holomorphic functions in the whole $\hat{\mathbb{C}} \setminus \{a_1, ..., a_r, \infty\}$. Near the points a_k, the roots $y_k(x)$ are bounded in terms of the coefficients of the polynomial in one variable $F(x, Y) \in \mathbb{C}[Y]$ and, similarly, $1/y_k(x)$ are bounded near ∞, so each function $s_i(x)$ extends to a meromorphic function defined in $\hat{\mathbb{C}}$, that can be written as a rational function $s_i(x) \in \mathbb{C}(x)$.

Consider now the polynomial $s(X)$, the least common multiple of the denominators of $s_i(X)$, and the polynomial

$$G(X, Y) = s(X)(Y^d - s_1(X)Y^{d-1} + s_2(X)Y^{d-1} - ... \pm s_d(X))$$

As any point $p \in W$ can be written as $(x, y_j(x))$ for some $j \in \{1, ..., d\}$,

$$\begin{aligned} G(p) &= s(x)(y_j^d(x) - s_1(x)y_j^{d-1}(x) + ... \pm s_d(x) \\ &= s(x)\prod_{i=1}^{d}(y_j(x) - y_i(x)) \\ &= 0 \end{aligned}$$

That is, $G(X, Y)$ and $F(X, Y)$ vanish in every point of W, so, by the Nullstelensatz, G is a multiple of F, and so, $\deg_Y(G) \geq \deg_Y(F)$, which means that $d = n$. From here is easy to see that, in fact, $F = G$ and so, $S_F^X = W$.

Obviously, the proof for S_F^Y is similar, and since S_F^X and S_F^Y coincide apart from finitely many points, by proposition 1.35 they have a common compactification $\hat{W} = S_F$. $\square$

Theorem 1.36 proves that every algebraic curve in the plane $F(X, Y) = 0$ gives a compact Riemann surface. Now we would like to know if every Riemann surface has the structure of an algebraic curve.

Recall that if S is a Riemann surface, we denote by $\mathcal{M}(S)$ the field of its meromorphic functions

$$\mathcal{M}(S) = \{f : S \to \mathbb{C} | f \text{ is meromorphic.}\}$$

We can see this field as a field extension $\mathbb{C}(f) \subset \mathcal{M}(S)$. In fact, the degree of this extension is given by the following proposition:

Proposition 1.37. *Let S be a Riemann surface, and let $f \in \mathcal{M}(S)$ be a degree n function. Then the field extension $\mathbb{C}(f) \subset \mathcal{M}(S)$ has degree $\leq n$.*

Proof. Let $h \in \mathcal{M}(S)$ be any function of the field. As f has degree n, it make sense to consider the points $y_1(x), ..., y_n(x) \in S$, the preimages of x by f, countend according with their multiplicities. Consider now the expressions

$$\begin{aligned} b_1(x) &= \sum h(y_i(x)) \\ b_2(x) &= \sum h(y_i(x))h(y_j(x)) \\ &\vdots \\ b_n(x) &= \prod h(y_i(x)) \end{aligned}$$

and set

$$p(y) = \prod(h(y) - h(y_i(f(y)))) = \sum(-1)^k b_k(f(y))h(y)^{n-k}$$

As in the previous theorem, the symmetric functions $b_i(x)$ define functions on the whole $\mathbb{P}^1$, and therefore, they are rational functions. On the other hand, as $y_i(x)$ is a preimage of x via f, $p(y)$ must vanish indentically.

In this conditions, we define the following polynomial in $\mathbb{C}(f)[Y]$:

$$P(Y) = Y^n - b_1(f)Y^{n-1} + ... \pm b_n(f) = \sum(-1)^k b_k(f)Y^{n-k}$$

The value of the function

$$P(h) = \sum(-1)^k b_k(f)h^{n-k}$$

at a point $y \in S$ is

$$P(h)(y) = p(y) = 0$$

We have found a polynomial of degree $\leq n$ with coefficients in $\mathbb{C}(f)$ such that every $h \in \mathcal{M}(S)$ satisfies it. Then by the Primitive Element Theorem, the extension has degree $\leq n$. □

Remark 1.38. By the Primitive Element Theorem, we can write $\mathcal{M}(S)$ as $\mathbb{C}(f, g)$ for some $g \in \mathcal{M}(S)$.

Now, we introduce a very important result in the theory of Riemann surfaces: The separation property of the field of meromorphic functions.

Theorem 1.39. *Let S be a compact Riemann surface of genus g. Then, the field of functions $\mathcal{M}(S)$ separates points, i.e, for every distinct $p, q \in S$ there is a meromorphic function $f \in \mathcal{M}(S)$ such that $f(p) \neq f(q)$. The field of functions also separates tangents, that is, for every point $p \in S$, there is a meromorphic function $f \in \mathcal{M}(S)$ such that f has a single pole at p.*

Proof. First, recall that the set of functions $\{f \in \mathcal{M}(S) | ord_p(f) \geq -n, ord_q(f) \geq 0 \, \forall q \neq p\}$ is in one-to-one correspondence with the space $H^0(L_p^n)$. This can be seen in the chapter about divisors in Miranda [3]. Fix two points p, q in S, and consider the line bundle $L = L_p^{g+1}$. By the Riemann-Roch Theorem 1.32, we have that

$$\dim H^0(L_p^{g+1}) \geq \deg(L) + 1(1-g) = g + 1 + (1-g) = 2$$

So we can find a non-constant function f suchh that $ord_p(f) \geq -n, ord_q(f) \geq 0 \, \forall q \neq p$. This function has to have a pole, and the only possibility is that the pole is in p. In particular, f has a pole in p and not in q and so, it separates p and q.

Using a similar argument, we can prove the separation of tangents. See Miranda [3] for the details.

□

Corollary 1.40. *Given two points p, q in a compact Riemann surface S, there exists a meromorphic function $\varphi \in \mathcal{M}(S)$ such that $\varphi(p) = 0$ and $\varphi(q) = \infty$.*

Remark 1.41. Some authors, for example Miranda [3], define Algebraic curves just as Riemann surfaces S where $\mathcal{M}(S)$ separates points and tangents. Anyway, the following theorem gives an idea of why the two definitions are equivalent.

Theorem 1.42. *Let $\mathcal{M}(S) = \mathbb{C}(f, h)$, and let $F(X, Y)$ be an irreducible polynomial such that $F(f, h)$ is identically zero. Then, the mapping*

$$S \xrightarrow{\Phi} S_F$$
$$P \mapsto (f(P), h(P))$$

defines an isomorphism.

Proof. Recall the notation of theorem 1.36. Let $x(S_F^X) = \hat{\mathbb{C}} \setminus \{a_1, ..., a_r, \infty\}$. Set $B = \{a_1, ..., a_r, \infty\}$ and $S^0 = S \setminus f^{-1}(B)$. In this conditions, the following diagram

$$\begin{tikzcd} S^0 \arrow[r, "\Phi"] \arrow[dr, "f"'] & S_F^X \arrow[d, "\mathbf{x}"] \\ & \hat{\mathbb{C}} \setminus B \end{tikzcd}$$

is commutative.
Note that if $f(P) = a \in \mathbb{C} \setminus \{a_1, ..., a_r, \infty\}$ then the value of h at P must be one of the n distinct roots of $F(a, Y)$, and thus, $\Phi(P)$ is a well defined point of S_F^X for every $P \in S^0$.

Now, we would like to use lemma 1.34 to extend Φ to the whole S. For doing that, we need to see that $\Phi : S^0 \to S_F^X$ is a covering map, but since x and f are, Φ is a covering map too. (See Theorem 1.74 in Girondo and González-Diez [5] for the details.)

It remains to show that Φ is an isomorphism, that is, that Φ has degree 1. Suppose not; then, the fibers of all but finitely many points $Q = (q_1, q_2) \in S_F^X$, would contain at least two points P_1, P_2. Let $\varphi \in \mathcal{M}(S)$. Since the field of functions $\mathcal{M}(S) = \mathbb{C}(f, h)$, the meromorphic function φ can be written as

$$\varphi = \frac{\sum a_{ij} f^i h^i}{\sum b_{ij} f^i h^j}$$

and so

$$\varphi(P_1) = \frac{\sum a_{ij} q_1^i q_2^j}{\sum b_{ij} q_1^i q_2^j} = \varphi(Q_2)$$

contradicting the separating property of the field of functions $\mathcal{M}(S)$ of corollary 1.40. $\square$

Corollary 1.43. *Let (F) be the ideal of $\mathbb{C}[X, Y]$ generated by F. Then:*

(i) The correspondence determined by $X \mapsto f$ and $Y \mapsto h$ defines a $\mathbb{C}$-isomorphism from the quotient field $\mathbb{C}[X,Y]/(F)$ to $\mathcal{M}(S)$.

(ii) The correspondence determined by $X \mapsto \mathbf{x}$, $Y \mapsto \mathbf{y}$ defines a $\mathbb{C}$-isomorphism from the quotient field $\mathbb{C}[X,Y]/(F)$ to $\mathcal{M}(S_F)$. In particular, $\mathcal{M}(S_F) = \mathbb{C}(\mathbf{x},\mathbf{y})$.

(iii) The polynomial $F(\mathbf{x},Y) \in \mathbb{C}(x)[Y]$ (resp. $F(f,Y) \in \mathbb{C}(f)[Y]$) is the minimal polynomial of $\mathbf{y}$ over $\mathbb{C}(\mathbf{x})$. (resp. h over $\mathbb{C}(f)$.

(iv) The degree of f is $\deg(f) = [\mathcal{M}(S) : \mathbb{C}(f)]$.

Proof. (i) Since $F(f,h) = 0 \in \mathcal{M}(S)$, the assignment $X \mapsto f$ and $Y \mapsto h$ defines a homomorphism

$$\rho : \mathbb{C}[X,Y] \to \mathcal{M}(S)$$

Lets study the kernel of ρ. Saying that $\rho(G(X,Y)) = 0$ means that $G(f,h) = 0$, which is equivalent to say that $G(X,Y)$ vanishes identically on the curve $F(x,y) = 0$ and, by Nullstellensatz, $G \in (F)$. So, by the first isomorphism theorem, we have (i).

(ii)Thanks to theorem 1.42, it is equivalent to (i).
(iii)Obvious.
(iv)The index $[\mathcal{M}(S) : \mathbb{C}(f)]$ is de degree of the minimal polynomial of h over $\mathbb{C}(f)$, namely $F(f,Y)$. This degree is $\deg_Y(F)$ which is equal to the degree of $\mathbf{x}$, which, by theorem 1.42, is the same as $\deg(f)$. □

With this theorems, we have shown the equivalence between the three categories we named in the begining of the chapter:

1) The category of Riemann surfaces.

2) The category of finite extensions of $\mathbb{C}(X)$.

3) The category of smooth projective algebraic curves

We pass from the first to the second by considering the functiont field $\mathcal{M}(S)$. We saw that this field can be seen as a finite field extension $\mathcal{M}(S) \subseteq \mathbb{C}(f)$, where f is a meromorphic function of degree n. By the Primitive Element Theorem, we can choose another generator $h \in \mathcal{M}(S)$ and an algebraic relation $F(f,h) = 0$ between them. This polynomial F will be our element in the third category: Algebraic curves. Finally, we saw in theorem 1.36 that we can give the structure of Riemann Surface to every Algebraic curve $F(X,Y)=0$.

Chapter 2

Belyi's Theorem

We know that we can associate to every irreducible polynomial $F(X,Y) \in K[X,Y]$ (where K is algebraically closed) a unique compact and connected Riemann surface S_F using the theorem 1.36.

Given a Riemann surface S, we say it is defined over a field $K \subset \mathbb{C}$ if S is isomorphic to a surface S_F, where F is a polynomial in $K[X,Y]$. This is the same as saying that there are two generators f, g of the function field $\mathcal{M}(S)$ such that $F(f,g) = 0$.

We are now interested in deciding whether or not a Riemann surface S is defined over a number field $K \subset \overline{\mathbb{Q}}$. With this purpose, we prove the following theorem, due to Belyi:

Theorem 2.1 (Belyi's Theorem)**.** *Let S be a compact Riemann surface. The following statements are equivalent:*

a) S is defined over a number field K, a finite extension of $\overline{\mathbb{Q}}$.

b) S admits a morphism $f : S \to \mathbb{P}^1$ with at most three branching values.

2.1 Proof of (a) $\implies$ (b)

Proposition 2.2. *Let $m, n \in \mathbb{N}$ and $\lambda = \frac{m}{m+n}$ and consider the following polynomial (Belyi's polynomial):*

$$P_{m,n}(x) = P_\lambda(x) = \frac{(m+n)^{m+n}}{m^m n^n} x^m (1-x)^n.$$

If we see Belyi's polynomial as a holomorphic map $P_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$, it satisfies:

(i) P_λ ramifies only at the points $x = 0, 1, \infty$ and λ.

(ii) $P_\lambda(0) = 0$, $P_\lambda(1) = 0$, $P_\lambda(\infty) = \infty$ *and* $P_\lambda(\lambda) = 1$.

Proof. The derivative of P_λ is

$$P'_\lambda = \frac{(m+n)^{m+n}}{m^m n^n}(x^{m-1}(1-x)^{n-1}(m-(m+n)x))$$

so the branching points of P_λ are $0, 1, \infty$ and λ, which proves (i).

The proof of (ii) is just a simple calculation. □

Definition 2.3 (Belyi function)**.** Let S be a compact Riemann surface. A morphism $f : S \to \mathbb{P}^1$ is called a **Belyi function** if it has only three branching values, namely $\{0, 1, \infty\}$

Remark 2.4. Some sources may define a Belyi function as a morphism with less than four branching values. The reason why we use this definition is the following:

- If f has no branching values, it is an unramified covering, hence an isomorphism between S and $\hat{\mathbb{C}}$.
- If f has one branching value $a \in \hat{\mathbb{C}}$, the restriction

$$f : S \setminus f^{-1}(a) \to \hat{\mathbb{C}} \setminus \{a\} \simeq \mathbb{C}$$

  is a unramified covering, hence an isomorphism between $S \setminus f^{-1}(a)$ and $\mathbb{C}$. It follows that $S \simeq \hat{\mathbb{C}}$.
- If f has two branching values, $a, b \in \hat{\mathbb{C}}$, the restriction

$$f : S \setminus f^{-1}(\{a, b\}) \to \hat{\mathbb{C}} \setminus \{a, b\}$$

  is an unramified covering. In fact, it is easy to see that this restriction must be isomorphic to

$$\begin{aligned} \mathbb{C} \setminus \{0\} &\longrightarrow \mathbb{C} \setminus \{0\} \\ z &\longmapsto z^n \end{aligned}$$

  for some k, so again, $S \simeq \mathbb{P}^1$.

Lemma 2.5. *Given a compact Riemann surface S and a morphism $f : S \longrightarrow \mathbb{P}^1$ ramified only over a set of rational values $\{0, 1, \infty, \lambda_1, ..., \lambda_n\} \subset \mathbb{Q} \cup \{\infty\}$, there exists a Belyi function defined in S.*

Proof. After composing f with the Möbius transformations $T(x) = 1 - x$ and $M(x) = 1/x$ (which permute $\{0, 1, \infty\}$), if necessary, we can asumme that $0 < \lambda_1 < 1$, so we can write $\lambda_1 = \frac{m}{m+n}$ for certain $m, n \in \mathbb{N}$. Consider the composition $P_{\lambda_1} \circ f$, which has as braching values $\{0, 1, \infty, P_{\lambda_1}(\lambda_2), ..., P_{\lambda_1}(\lambda_n)\}$.

Now, inductively, we can find the desired morphism f' with at most, three branching values. $\square$

Now, we are in position to prove the first part of the Belyi's Theorem, $(a) \Rightarrow (b)$.

Theorem 2.6. *Let S be a compact Riemann surface. If S is defined over a number field K, a finite extension of $\overline{\mathbb{Q}}$, them it admits a morphism $f : S \to \mathbb{P}^1$ with at most three branching values.*

Proof. Let us write the Riemann surface S as S_F with

$$F(X, Y) = p_0(X)Y^n + p_1(X)Y^{n-1} + ... + p_n(X) \in \overline{\mathbb{Q}}[X, Y]$$

and consider the morphism given by

$$\begin{aligned} S_F &\xrightarrow{\mathbf{x}} \mathbb{P}^1 \\ (x, y) &\longmapsto x \end{aligned}$$

Denote by $B_0 = \{\mu_1, ..., \mu_s\}$ the set of branching values of $\mathbf{x}$. By the definitions on theorem 1.36, each μ_i is either a zero of $p_0(X)$ or the point $\infty \in \mathbb{P}^1$, or the first coordinate of a common zero of F and F_Y so, by Bezout's theorem, $B_0 \subset \overline{\mathbb{Q}} \cup \{\infty\}$.
If $B_0 \subset \mathbb{Q} \cup \{\infty\}$, we have found a morphism with a finite set of branching values, all contained in $\mathbb{Q} \cup \{\infty\}$, so, by lemma 2.5, we have finished the proof.
If not, let $m_1(T)$ be the minimal polynomial of $\mu_1, ..., \mu_s$ in $\mathbb{Q}$. Let $\beta_1, ..., \beta_d$ be the roots of the derivative $m_1'(T)$ and $p(T)$ their minimal polynomial in $\mathbb{Q}$.

Since the following identity holds for the branch values of any twho holomorphic functions f, g,

$$Branch(g \circ f) = Branch(g) \cup g(Branch(f))$$

we have that the set of branching values of the composition

$$\begin{aligned} m_1 \circ \mathbf{x} : S_F \xrightarrow{\mathbf{x}} \mathbb{P}^1 &\xrightarrow{m_1} \mathbb{P}^1 \\ (x, y) \mapsto x &\mapsto m_1(x) \end{aligned}$$

is

$$B_1 = m_1(\{\beta_1, ..., \beta_d\}) \cup \{0, \infty\}$$

If $B_1 \subset \mathbb{Q} \cup \{\infty\}$, we are done. If not, we continue the process by denoting by $m_2(T)$ the minimal polynomial of the branching values of m_1, which are $m_1(\{\beta_1, ..., \beta_d\})$. Clearly, $[\mathbb{Q}(m_1(\beta_i)) : \mathbb{Q}] \leq [\mathbb{Q}(\beta_i) : \mathbb{Q}]$, so the degree of the minimal polynomial of $m_1(\beta)$ is lower or equal to the degree of the minimal polynomial of β_i. If β_i and β_j had the same minimal polynomial, there would be a field embedding $\sigma : \mathbb{Q}(\beta_i) \to \overline{\mathbb{Q}}$ such that $\sigma(\beta_i) = \beta_j$) but in that case, $\sigma(m_1(\beta_i)) = m_1(\beta_j)$ and so, $m_1(\beta_i)$ and $m_1(\beta_j)$ have also the same minimal polynomial. Therefore,

$$\deg(m_2(T)) \leq \deg(p(T)) \leq \deg(m_1'(T)) < \deg(m_1(T)) \tag{2.1}$$

The set of branching values of the composition $m_2 \circ m_1 \circ \mathbf{x}$ is

$$B_2 = m_2(\{\text{roots of } m_2'\}) \cup m_2(B_1)$$

We know that $m_2(B_1) = \{0, \infty, m_2(0)\}$, so $m_2(B_1) \subset \mathbb{Q} \cup \{\infty\}$. If the whole B_2 is in $\mathbb{Q} \cup \{\infty\}$, we are done. If not, we continue the process denoting by $m_3(T)$ the minimal polynomial of $m_3(\{\text{roots of } m_3'\})$ and looking at the set B_3 of branching values of $m_3 \circ m_2 \circ m_1 \circ \mathbf{x}$, which is, of course,

$$B_3 = m_3(\{\text{roots of } m_3'\}) \cup m_3(B_2)$$

This process must end after finitely many steps, as by equation 2.1, we have that

$$\deg(m_i(T)) < \deg(m_{i+1}(T))$$

and we would have $B_k \subset \mathbb{Q} \cup \{\infty\}$, and thus, a morphism in the conditions of lemma 2.5, which gives us the result. □

2.2 Proof of (b) $\implies$ (a)

Before proving this implication, we need some previous notions of morphisms between Riemann surfaces and some theory of Galois actions.

2.2.1 Morphisms between Riemann surfaces

Proposition 2.7. *Defining a morphism $f : S_F \to S_G$ is equivalent to giving a pair of rational functions*

$$f = (R_1(X,Y), R_2(X,Y)) = \left(\frac{P_1(X,Y)}{Q_1(X,Y)}, \frac{P_2(X,Y)}{Q_2(X,Y)}\right)$$

with $Q_i \notin (F)$, such that

$$Q_1^n Q_2^m G(R_1, R_2) = HF$$

where $n = \deg_X G$, $m = \deg_Y G$ and $H \in \mathbb{C}[X,Y]$.

Proof. Consider the identity

$$G(R_1(x,y), R_2(x,y)) = 0$$

If we clear denominators, we obtain

$$Q_1^n(x,y)Q_2^m(x,y)G(R_1(x,y), R_2(x,y)) = 0$$

which, by the Nullstelensatz, implies that F divides $Q_1^n(x,y)Q_2^m(x,y)G(R_1(x,y), R_2(x,y))$, or which is the same,

$$Q_1^n Q_2^m G(R_1, R_2) = HF$$

□

Now, we would like to know when a morphism, defined in the above way, is a isomorphism. This occurs, of course, if we have a morphism $h : S_G \to S_F$ inverse to f. By the proposition 2.7, this means that there are two rational functions $W_i = \frac{U_i}{V_i}$, $i = 1, 2$, with $V_i \notin (G)$, satisfying

$$V_1^s V_2^t F(W_1, W_2) = TG$$

Moreover, the morphism h has to satisfy that

$$\begin{aligned} h \circ f(x,y) &= h(R_1(x,y), R_2(x,y)) = \\ &= \left(\frac{U_1(R_1(x,y), R_2(x,y))}{V_1(R_1(x,y), R_2(x,y))}, \frac{U_2(R_1(x,y), R_2(x,y))}{V_2(R_1(x,y), R_2(x,y))}\right) = \\ &= (x,y) \end{aligned}$$

Clearing out denominators and, again, using Nullstelensatz, we see that this is satisfied if and only if these two identities are satisfied:

$$Q_1^d Q_2^k (U_1(R_1, R_2) - XV_1(R_1, R_2)) = H_1 F \tag{2.2}$$

$$Q_1^d Q_2^k (U_2(R_1, R_2) - YV_2(R_1, R_2)) = H_2 F \tag{2.3}$$

where $d = deg_X(U_i - XV_i)$ and $k = deg_Y(U_i - YV_i)$.

This characterizes in an algebraic way the isomorphisms between Riemann surfaces.

2.2.2 Discrete valuations and the Galois action

Consider the (huge) group $\mathrm{Gal}(\mathbb{C}) = \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ of all field automorphisms of $\mathbb{C}$ that fix the field $\mathbb{Q}$.

Definition 2.8. For a given $\sigma \in \mathrm{Gal}(\mathbb{C})$ and a complex number $a \in \mathbb{C}$, we define a^σ to be $\sigma(a)$. Accordingly, we shall employ the following notation:

(i) For a polynomial $P = \sum a_{ij} X^i Y^j \in \mathbb{C}[X, Y]$, we will write $P^\sigma = \sum a_{ij}^\sigma X^i Y^j$.

(ii) If $R(X, Y) = \frac{P(X,Y)}{Q(X,Y)}$ is a rational function, we will write $R^\sigma = \frac{P^\sigma}{Q^\sigma}$.

(iii) If $S \simeq S_F$, we write $S^\sigma \simeq S_{F^\sigma}$.

(iv) If $\Psi : S_F \to S_G$ is a morphism given by $\Psi = (R_1, R_2)$, we define $\Psi^\sigma : S_F^\sigma \to S_G^\sigma$ to be $\Psi^\sigma = (R_1^\sigma, R_2^\sigma)$.

Definition 2.9. Let $\mathcal{M}$ be a function field, that is, a field that is $\mathbb{C}$-isomorphic to a finite extension of the field of rational functions $\mathbb{C}(X)$. Consider the multiplicative group $\mathcal{M}^* = \mathcal{M} \setminus \{0\}$. A **discrete valuation** of $\mathcal{M}$ is a map

$$\upsilon : \mathcal{M}^* \longrightarrow \mathbb{Z}$$

such that:

1. υ is a group homomorphism.
2. $\upsilon(\varphi \pm \psi) \geq \min\{\upsilon(\varphi), \upsilon(\psi)\}$, the equality holding whenever $\upsilon(\varphi) \neq \upsilon(\psi)$.
3. $\upsilon(\varphi) = 0$ if $\varphi \in \mathbb{C}^*$.
4. υ is non-trivial.

Remark 2.10. We can extend a valuation $\upsilon : \mathcal{M}^* \longrightarrow \mathbb{Z}$ to the whole $\mathcal{M}$ by setting $\upsilon(0) = +\infty$.

Remark 2.11. The following hold:

- $A_\upsilon = \{\varphi \in \mathcal{M} : \upsilon(\varphi) \geq 0\}$ is a subring of $\mathcal{M}$.
- $\varphi \in A_\upsilon$ is a unit $\Leftrightarrow \upsilon(\varphi) = 0$.
- The set $M_\upsilon = \{\varphi \in A_\upsilon : \upsilon(\varphi) > 0\}$ of all non-units of A_υ forms a maximal ideal of A_υ.
- If $\upsilon(\mathcal{M}^*) = (m_\upsilon)$ for some $m_\upsilon \in \mathbb{Z}$, then $M_\upsilon = (\varphi)$ if and only if $\upsilon(\varphi) = m_\upsilon$. We say in this case that A_υ is *a local ring with maximal ideal M_υ and uniformizing parameter φ.*

It is usual to assume that υ is surjective, which can be achieved by defining the **normalization** of υ as $\upsilon^*(\varphi) := \frac{\upsilon(\varphi)}{m_\upsilon}$.

Every point P of a compact Riemann surface S defines a valuation υ_P on the field $\mathcal{M}(S)$ by the formula

$$\upsilon_P(\varphi) = \mathrm{ord}_P(\varphi).$$

Lemma 2.12. *Let υ_1, υ_2 be two normalized valuations of $\mathcal{M}$. Then $\upsilon_1 = \upsilon_2$ if and only if $A_{\upsilon_1} = A_{\upsilon_2}$.*

Proof. If $A_{\upsilon_1} = A_{\upsilon_2}$ then $M_{\upsilon_1} = M_{\upsilon_2} = (\varphi)$, that is, $\upsilon_1(\varphi) = \upsilon_2(\varphi) = 1$. Now, for $\psi \in \mathcal{M}^*$, we have that

$$\upsilon_1(\psi) = n \leftrightarrow \upsilon_1(\frac{\varphi^n}{\psi}) = 0$$

which happens if and only if $\frac{\varphi^n}{\psi}$ is a unit of A_{υ_1}, that is, $\frac{\varphi^n}{\psi}$ is a unit of A_{υ_2}, which again, is equivalent to $\upsilon_1(\psi) = n$. □

Proposition 2.13. *For any pair of distinct valuations υ_1, υ_2 of $\mathcal{M}$ there exists an element $\varphi \in \mathcal{M}$ such that $\upsilon_1(\varphi) \geq 0$ and $\upsilon_2(\varphi) < 0$.*

Proof. See proposition 3.19 in Girondo and González-Diez [5]. □

Definition 2.14 (Galois action on points)**.** Let $\sigma \in \mathrm{Gal}(\mathbb{C})$.

- Given a valuation υ on $\mathcal{M}(S)$, we define the valuation υ^σ on $\mathcal{M}(S^\sigma)$ to be

$$\upsilon^\sigma = \upsilon \circ \sigma^{-1}$$

- Accordingly, for a point $P \in S$ we define P^σ to be the only point of S^σ such that $\upsilon_{P^\sigma} = (\upsilon_P)^\sigma$.

Definition 2.15. A Belyi Pair (S, f) consists of a Compact Riemann Surface together with a Belyi function, that is, a map

$$f : S \to \mathbb{C}$$

whose only branch values are $\{0, 1, \infty\}$.

The following theorem describes the Galois action on Belyi Pairs.

Theorem 2.16. *The action of* $\mathrm{Gal}(\mathbb{C})$ *on pairs* (S, f) *enjoys the following properties:*

(1) $\deg(f^\sigma) = \deg(f)$

(2) $(f(P))^\sigma = f^\sigma(P^\sigma)$

(3) $\mathrm{ord}_{P^\sigma}(f^\sigma) = \mathrm{ord}_P(f)$

(4) $a \in \hat{\mathbb{C}}$ *is a branching value of* f *if and only if* a^σ *is a branching value of* f^σ.

(5) The genus of S^σ *is the same as the genus of* S*, that is,* S *and* S^σ *are homeomorphic.*

(6) The rule

$$\begin{aligned} \mathrm{Aut}(S, f) &\longrightarrow \mathrm{Aut}(S^\sigma, f^\sigma) \\ h &\longmapsto h^\sigma \end{aligned}$$

is a groups isomorphism.

(7) The monodromy group $\mathrm{Mon}(f)$ *of the covering* (S, f) *is isomorphic to the monodromy* $\mathrm{Mon}(f^\sigma)$ *of the covering* (S^σ, f^σ).

Proof. See Theorem 3.28 in Girondo and González-Diez [5]. □

2.2.3 A criterion for definability over $\overline{\mathbb{Q}}$

The aim of these section is to prove the following theorem, that will help us to finish the proof of Belyi's theorem. With this purpose, we need to introduce the notion of *specialization* and *infinitesimal specialization.*

Theorem 2.17. *For a compact Riemann surface S, the following conditions are equivalent:*

1. *S is defined over $\overline{\mathbb{Q}}$.*
2. *The family $\{S^\sigma\}_{\sigma\in\mathrm{Gal}(\mathbb{C})}$ contains only finitely many isomorphism classes of Riemann surfaces.*

Definition 2.18. Let k be a subfield of $\mathbb{C}$. A finite set of complex numbers $\{\pi_1, ..., \pi_d\} \subset \mathbb{C}$ is said to be **algebraically independent over** k if the evaluation map is injective, that is, it induces an isomorphism between $k[X_1, ..., X_d]$ and $k[\pi_1, ..., \pi_d]$.

In this conditions, a d-tuple $(q_1, ..., q_d) \in \mathbb{C}^d$ induces a well defined homomorphism

$$\begin{aligned} \mathbf{s} : k[\pi_1, ..., \pi_d] &\longrightarrow \mathbb{C} \\ a(\pi_1, ..., \pi_d) &\longmapsto a(q_1, ..., q_d) \end{aligned}$$

By a **specialization** of $(\pi_1, ..., \pi_d)$ we mean either a d-tuple $(q_1, ..., q_d) \in \mathbb{C}^d$ or the map

$$\mathbf{s} : k[\pi_1, ..., \pi_d] \longrightarrow \mathbb{C}$$

and the **distance** of the specialization will be $\max_i \mid \pi_i - q_i \mid$.

Note that a finite set of complex numbers $\pi_1, ..., \pi_d$ is algebraically independent over k if and only if each π_i is trascendental over the field $k(\pi_1, ..., \pi_{i-1})$.

A field extension K of k is called *purely trascendental* if it is generated over k by a set of algebraically independent elements over k.

From now, we will denote by $(\pi_1, ..., \pi_d, u)$ a $(n+1)$-tuple such that $\pi_1, ..., \pi_d$ are algebraically independent and u is algebraic over $\mathbb{Q}(\pi_1, ..., \pi_d)$. If $(q_1, ..., q_d)$ is a specialization of $(\pi_1, ..., \pi_d)$, consider the $\mathbb{Q}$-algebra homomorphism $\mathbf{s}$. We would like to extend $\mathbf{s}$ to the whole ring $\mathbb{Q}[\pi_1, ..., \pi_d, u]$. In order to do that, we introduce the following notation:

$$a^s = s(a) = a(q_1, ..., q_d) \text{ for an element } a \in \mathbb{Q}[\pi_1, ..., \pi_d]$$

and

$$q^s(X) = \sum a_l^s X^l$$

for a polynomial $q(X) = \sum a_l X^l \in \mathbb{Q}[\pi_1, ..., \pi_d][X]$.

In order to extend $\mathbf{s}$, consider the minimal polynomial of u over $\mathbb{Q}[\pi_1, ..., \pi_d]$

$$m_u(X) = \sum a_j X^j$$

Then, we have

$$0 = s(0) = s(m_u(u)) = m_u^s(s(u))$$

so $s(u)$ must be a root of the polynomial $m_u^s(X)$. Moreover,

Lemma 2.19. *With this notation, let b be any root of $m_u^s(X)$. Then the assignment*

$$\pi_i \mapsto \mathrm{q}_i$$
$$u \mapsto b$$

extends $\boldsymbol{s}$.

Lemma 2.20. *With above's notation, let $u = u_1, u_2, ..., u_n \in \mathbb{C}$ be the roots of $m_u(X)$, and let $\delta = \min_{k,l} \mid u_k - u_l \mid$. There is a real positive number $\epsilon(u)$ such that if $\boldsymbol{s}$ is the homomorphism determined by a specialization of distance less than $\epsilon(u)$, then the polynomial $m_u^s(X)$ has a unique root u_s with the property that $\mid u - u_s \mid < \delta$.*

Definition 2.21. An **infinitesimal specialization** of $(\pi_1, ..., \pi_d; u)$ is a specialization of $(\pi_1, ..., \pi_d)$ of distance less than $\epsilon(u)$ in the conditions of lemma 2.20.

We will call **homomorphism associated to an infinitesimal specialization** (or simply, infinitesimal specialization) of $(\pi_1, ..., \pi_d, u)$ to the $\mathbb{Q}$-algebra homomorphism $\mathbf{s}$.

$$\mathbf{s} : k[\pi_1, ..., \pi_d] \longrightarrow \mathbb{C}$$

. Now, we are in position to prove the criterion 2.17.

Proof. Clearly, if S is defined over $\overline{\mathbb{Q}}$, that is, if $S = S_F$ for some polynomial F with coefficients in some finite Galois extension K of $\mathbb{Q}$, then the family $\{F^\sigma\}_{\sigma \in \mathrm{Gal}(\mathbb{C})}$ consists of, at most, $[K : \mathbb{Q}]$ different polynomials. In particular, the family $\{S^\sigma\}_{\sigma \in \mathrm{Gal}(\mathbb{C})}$ contains only finitely many Riemann surfaces, hence finitely many isomorphism classes of Riemann surfaces.

Reciprocally, let $\Sigma_1 := \{\pi_1, ..., \pi_d\}$ be a maximal set of algebraically independent coefficients of $F = F(X, Y)$. By the Primitive Element Theorem, the field K_1 generated by all coefficients of F is

$$K_1 = \mathbb{Q}(\pi_1, ..., \pi_d, \upsilon)$$

with υ algebraic over $\mathbb{Q}(\pi_1, ..., \pi_d)$, and with minimal polynomial $m_\upsilon(T)$. For any $\sigma \in \mathrm{Gal}(\mathbb{C})$, the field generated by the coefficients of F^σ will be

$$K_2 = \sigma(K_1) = \mathbb{Q}(\sigma(\pi_1), ..., \sigma(\pi_d), \sigma(\upsilon))$$

Now, consider $\sigma \in \text{Gal}(\mathbb{C})$ such that the set

$$\Sigma_2 := \{\pi_1, ..., \pi_d, \pi_{d+1} := \sigma(\pi_1), ..., \pi_{2d} := \sigma(\pi_d)\}$$

is a set of algebraically independient elements. As the family $\{S^\sigma\}_{\sigma \in \text{Gal}(\mathbb{C})}$ contains only finitely many isomorphism classes, there are plenty of pairs $\beta, \tau \in \text{Gal}(\mathbb{C})$ such that there is an isomorphism $\Phi : S_{F^\tau} \to S_{F^\beta}$, hence an isomorphism $\Psi = \Phi^{\tau^{-1}} : S_F \to S_{F^\sigma}$ with $\sigma = \tau^{-1} \circ \beta$.

Now we remember the characterization that we did of isomorphisms in proposition 2.7 and equations 2.2 and 2.3, and we enlarge the set Σ_2 by adding some coefficients of the polynomials $P_i, Q_i, U_i, V_i, T, H_i$ and H that appeared in those equations, namely, we add a maximal possible collection of that coefficients with the condition that the set $\Sigma_3 = \{\pi_1, ..., \pi_n\}$ is still an algebraically independent set. The field K_3 generated by K_1, K_2 and the whole set of the coefficients of these polynomials is

$$K_3 = \mathbb{Q}(\pi_1, ..., \pi_n, u)$$

where u is algebraic over $\mathbb{Q}(\pi_1, ..., \pi_n)$.

For $j = d+1, ..., n$, consider $q_j \in \mathbb{Q}(i)$ such that $(\pi_1, ..., \pi_d, q_{d+1}, q_n)$ is an infinitesimal specialization of $(\pi_1, ..., \pi_n, u)$, and let $\mathbf{s}$ denote the associated homomorphism.

Consider the elements z of $\mathbb{Q}(\pi_1, ..., \pi_n, u)$, the field of fractions of the ring $\mathbb{Q}[\pi_1, ..., \pi_n, u)]$, and consider the subring $\mathbb{Q}[\pi_1, ..., \pi_n, u]_s$ of elements $z = \frac{A(\pi_1,...,\pi_n,u)}{B(\pi_1,...,\pi_n,u)} \in \mathbb{Q}(\pi_1, ..., \pi_n, u)$ such that $\mathbf{s}(B) \neq 0$.

Clearly, $\mathbf{s}$ extends to a unique homomorphism $\mathbf{s} : \mathbb{Q}[\pi_1, ..., \pi_n, u]_s \to \mathbb{C}$ and hence, to a homomorphism $\mathbf{s} : \mathbb{Q}[\pi_1, ..., \pi_n, u]_s[X, Y] \to \mathbb{C}[X, Y]$.

Now, if the distance of our specialization is small enough, all coefficients of the polynomials $P_i, Q_i, U_i, V_i, T, H_i$ and H and the element $\upsilon \in K_1$ lie in $\mathbb{Q}[\pi_1, ..., \pi_n, u]_s$. Under these conditions, we can apply $\mathbf{s}$ to the identities 2.2 , 2.3 and the one in proposition 2.7, getting an isomorphism

$$\Psi^s = S_{F^s} \to S_{(F^\sigma)^s)}$$

Then, the coefficients of $F^\sigma)^s$ must be in the field generated by the numbers $q_j \in \mathbb{Q}(i)$, and $\mathbf{s}(\sigma(\upsilon))$, which must be a root of the polynomial $(m_\upsilon^\sigma)^s(X) \in \mathbb{Q}(i)[X]$, and therefore, an algebraic number, so $F^\sigma)^s$ has coefficients in a number field.

It only remains to prove that $F^s = F$. Because of how we constructed $\mathbf{s}$, it is enough to see that $s(\upsilon) = \upsilon$. Clearly, $s(\upsilon)$ is a root of m_υ^s, and by lemma 2.20, if we get the distance of our specialization small enough, then m_υ^s has a unique root, and so, $s(\upsilon) = \upsilon$. $\square$

2.2.4 Ending the proof

With the criterion 2.17, it is quite easy to finish the proof of Belyi's theorem:

$(b) \Rightarrow (a)$. If $f : S \to \mathbb{P}^1$ is a morphism of degree d whose only branching values are $0, 1$ and ∞, for any Galois element $\sigma \in \mathrm{Gal}(\mathbb{C})$, the morphism $f^\sigma : S^\sigma \to \mathbb{P}^1$ is, by theorem 2.16 a morphism of degree d having the same branching values. As the group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ is the free group with two generators, the family $\{f^\sigma\}$ gives rise to only finitely many different monodromy[1] homomorphisms

$$M_{f^\sigma} : \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to \Sigma_d$$

Two morphisms $f_i : S_i \to S$ of the same degree and the same branching values have conjugated monodromies if and only if they are isomorphic (as coverings)(and thus, S_1 and S_2 are isomorphic),so we have that the family $\{S^\sigma\}_{\sigma \in \mathrm{Gal}(\mathbb{C})}$ must have only finitely many isomorphism classes, and so, by the criterion of theorem 2.17, the surface S is defined over a number field. □

[1]We will talk of monodromy in the next chapter, but for a reference go to section 2.7 inGirondo and González-Diez [5]

Chapter 3

Dessins d'enfants

Although the concept of Dessin d'enfants was already used by Felix Klein in 1879 to construct a 11-fold cover of $\mathbb{P}^1$ by itself with monodromy group $\mathrm{PSL}(2, \mathbb{Z}_{11})$[1], the modern way we understand the "Children's drawings" was discovered by Alexander Grothendieck while studying the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
There are many ways to define dessins d'enfants. The first one is to consider a compact orientable surface, and draw a graph (with some conditions) on it. With some effort, this graph will provide the surface with the structure of a covering of the Riemann sphere, ramified only over the values $\{0, 1, \infty\}$.

Definition 3.1. A **dessin d'enfants** is a compact oriented topological surface X with a graph $\mathcal{D}$ embedded in it, such that:

1. The graph $\mathcal{D}$ is connected.
2. The graph $\mathcal{D}$ is bicoloured, that is, each vertex is assigned one of two colours, black or white, and two vertices connected by an edge must have different colours.
3. Each connected component of $X \setminus \mathcal{D}$ is homeomorphic to a topological disk. The connected components of $X \setminus \mathcal{D}$ will be called faces of the dessin.

The genus of a dessin is the genus of the surface X.

Example 3.1. *Any tree contained in the sphere is a dessin with one face, as a tree can always be bicoloured.*

Definition 3.2. Two dessins $(X, \mathcal{D})$ and $(X', \mathcal{D}')$ are equivalent if there exists an orientation-preserving homeomorphism $\Phi : X \to X'$ whose restriction to $\mathcal{D}$ induces an isomorphism between $\mathcal{D}$ and $\mathcal{D}'$.

[1]Klein used something similar to Dessins to construct this cover in Klein [6]

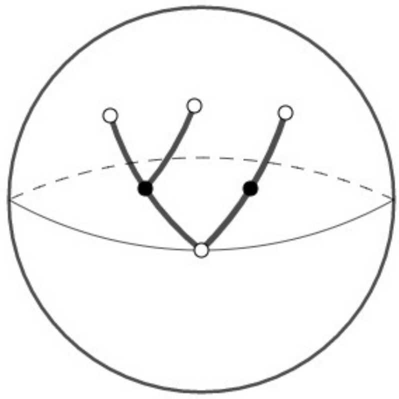


FIGURE 3.1: A dessin over the sphere

Remark 3.3. A dessin is more than an absract graph: it comes equipped with an embedding into the surface X. For example, the graphs of both dessins in next figure coincide, but the dessins are not equivalent, as they do not have the same genus.

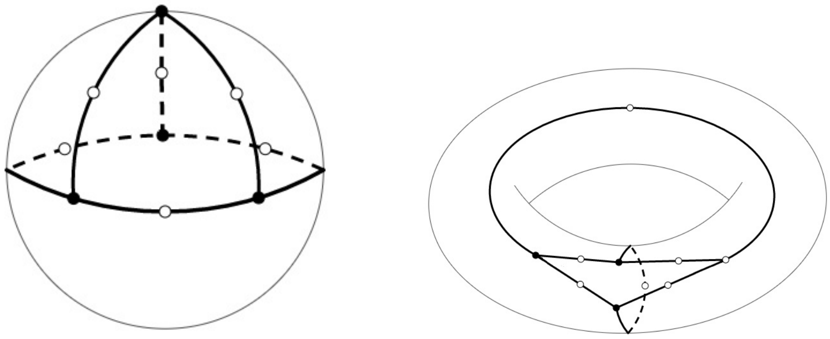


FIGURE 3.2: Two dessins with the same underlying graph

3.1 Dessins d'enfants and Belyi pairs

Consider now a dessin d'enfants $(X, \mathcal{D})$. Choose a point in the interior of each face of the dessin, and a direction to walk through every face of the dessin. Every time you pass through one of the vertex, join it to the "center" of the face, dividing the surface in triangles. As the graph is bicolored, we can choose one color, black or white, for each of these triangles in a way that each edge j of the graph has one side black, the triangle T_j^- and one side white, the triangle T_j^+.

Using these triangles, we can define a covering map from X to the Riemann sphere $\mathbb{P}^1$ by mapping the white triangles to the north hemisphere, the black triangles to the south one, the white points to $0 \in \mathbb{P}^1$, the black points to $1 \in \mathbb{P}^1$ and the center of each face to ∞. For example, we can triangulate the torus with the following graph embedded in it using 8 triangles to produce a 4-sheeted (branched) covering of the Riemann sphere:

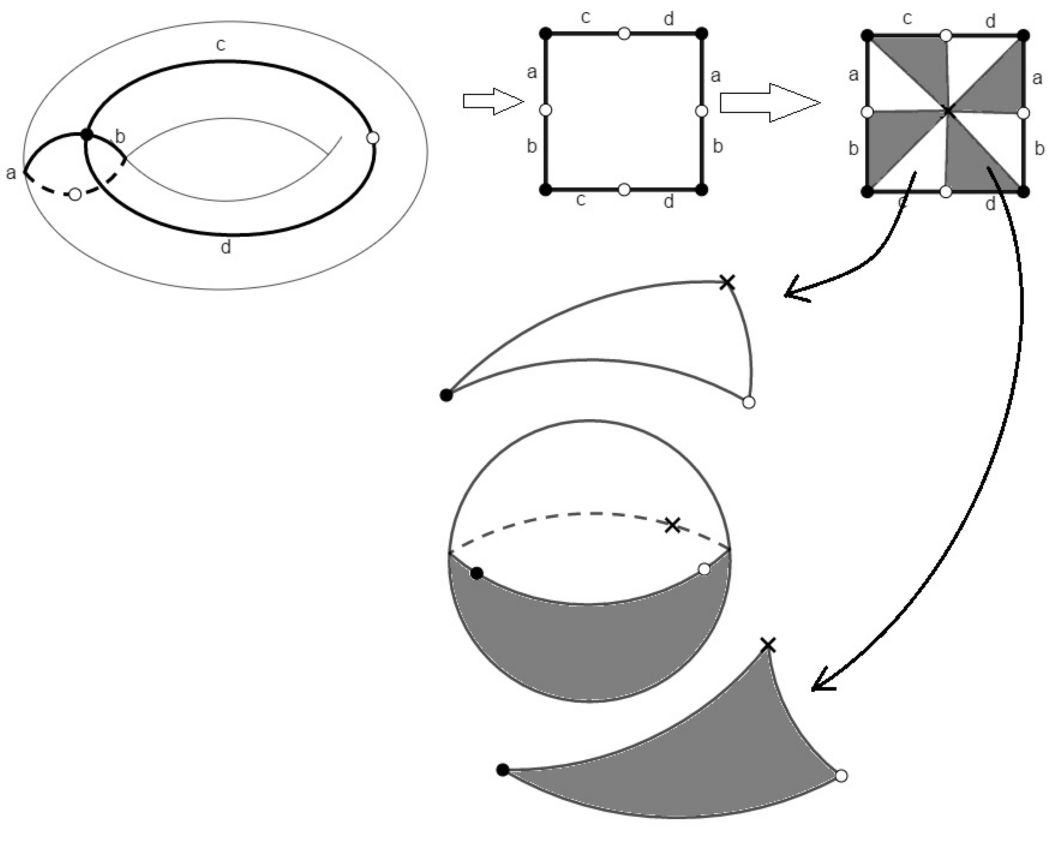


FIGURE 3.3: An example with the torus.

Remark 3.4. For each triangle $T_j^{\pm}$, we have chosen a homeomorphism $f_j^{\pm}$ to the hemispheres of the Riemann sphere. We can glue together these homeomorphims to construct a continuous function $f_{\mathcal{D}} : X \to \mathbb{P}^1$ whose restriction to $X^* = X \setminus f_{\mathcal{D}}^{-1}(\{0,1,\infty\}$

$$f_{\mathcal{D}} : X^* \to \mathbb{P}^1 \setminus \{0,1,\infty\}$$

is a topological covering. By the lemma 1.34, we can then provide X^* with the only Riemann surface structure that makes $f_{\mathcal{D}}$ holomorphic, which makes X a compact Riemann surface denoted by $S_{\mathcal{D}}$.

With this procedure, we have obtained a map $f_{\mathcal{D}}$ from X to $\mathbb{P}^1$ with the following properties:

1. The map $f_{\mathcal{D}}$ ramifies only at the vertices of the triangles.
2. In particular, the only branching values of $f_{\mathcal{D}}$ are 0, 1 and ∞, so $f_{\mathcal{D}}$ is a Belyi function.
3. $\deg(f_{\mathcal{D}})$ agrees with the number of edges of $\mathcal{D}$, as can be seen counting the preimages of any real value between 0 and 1.ç
4. The multiplicity of $f_{\mathcal{D}}$ at the center of a face is half of the number of edges of the face.
5. $f_{\mathcal{D}}^{-1}([0,1]) = \mathcal{D}$

Definition 3.5. Recall that we call **Belyi pair** to a pair (S,f) where S is a Compact Riemann Surface and f is a Belyi function. We say that two Belyi pairs (S_1,f_1) and (S_2,f_2) are equivalent if there exist a morphism $\phi : S_1 \to S_2$ such that the following diagram commutes:

$$\begin{tikzcd} S_1 \arrow[rr, "\phi"] \arrow[dr, "f_1"'] & & S_2 \arrow[dl, "f_2"] \\ & \hat{\mathbb{C}} & \end{tikzcd}$$

The pair $(S_{\mathcal{D}}, f_{\mathcal{D}})$ constructed above is called the *Belyi pair associated to* $(X, \mathcal{D})$.

Remark 3.6. The rule

$$\{Dessins\} \longrightarrow \{Belyi\ pairs\}$$

which sends $(X,\mathcal{D})$ to $(S_{\mathcal{D}}, f_{\mathcal{D}})$ induces a well defined map from equivalence classes of Dessins d'enfants to equivalence classes of Belyi pairs.

Conversely, we have the following proposition:

Proposition 3.7. *Let (S, f) be a Belyi pair, and consider the bicoloured graph $\mathcal{D}_f$ given by $f^{-1}([0,1])$, where the white vertices are $f^{-1}(0)$ and the black vertices are $f^{-1}(1)$. Then:*

1. *$\mathcal{D}_f$ is a dessin d'enfants.*
2. *Each of the sets $f^{-1}([-\infty, 0])$, $f^{-1}([0,1])$ and $f^{-1}([1,\infty])$ is a union of topological segments, namely the set of edges of a triangle descomposition of $\mathcal{D}$.*
3. *$f = f_{\mathcal{D}_f}$.*

Proof. See Proposition 4.22 in Girondo and González-Diez [5]. □

Corollary 3.8. *There is a one-to-one correspondence between Dessins and Belyu pairs.*

Remark 3.9. This proposition gives us the correspondence between equivalence classes of Dessins d'enfants and equivalence classes of Belyi pairs, since if two Belyi pairs (S_1, f_1) and (S_2, f_2) are equivalent, then there is an isomorphism $\tau : S_1 \to S_2$, and the homeomorphism induced by τ on the surfaces (seen as topological surfaces) gives us an equivalence of the corresponding dessins $\mathcal{D}_1 = f_1^{-1}([0,1])$ and $\mathcal{D}_2 = f_2^{-1}([0,1]) = \tau^{-1}(\mathcal{D}_1)$.

Definition 3.10. A morphism between the dessin d'enfants $(S_1, \mathcal{D}_1)$ and $(S_2, \mathcal{D}_2)$ is a morphism $\Phi : S_1 \to S_2$ such that the diagram

$$\begin{tikzcd} S_1 \arrow[r, "f_{\mathcal{D}_1}"] \arrow[d, "\Phi"] & \mathbb{P}^1 \arrow[d, "Id"] \\ S_2 \arrow[r, "f_{\mathcal{D}_2}"] & \mathbb{P}^1 \end{tikzcd}$$

commutes.

One of the consequences of this characterization is that the genus of the surface X is encoded in the graph $\mathcal{D}$ (and its embedding to X):

Proposition 3.11. *Let $(X, \mathcal{D})$ be a dessin with v vertices, e edges and f faces. Let g be the genus of X. Then, Euler formula:*

$$2 - 2g = \chi(X) = v - e + f$$

holds.

Proof. Remember the Riemann-Hurwitz formula for a morphism $f : S_1 \to S_2$:

$$-\chi(S_1) = \deg(f)(-\chi(S_2)) + \sum_{x \in S_1} (m_x(f) - 1)$$

If we apply it to the Belyi map $f_{\mathcal{D}}$ we obtain:

$$-\chi(X) = \deg(f_{\mathcal{D}})(-2) + \sum_{P \in X} (m_P(f_{\mathcal{D}}) - 1) = -2e + b_0 + b_1 + b_\infty$$

where $b_0 = \sum_{f_{\mathcal{D}}(P)=0}(m_P(f_{\mathcal{D}})-1)$, $b_1 = \sum_{f_{\mathcal{D}}(P)=1}(m_P(f_{\mathcal{D}})-1)$ and $b_\infty = \sum_{f_{\mathcal{D}}(P)=\infty}(m_P(f_{\mathcal{D}})-1)$. Now, as the multiplicity of $f_{\mathcal{D}}$ at the preimages of 0, 1, and ∞ is encoded in the degree of the vertices of $\mathcal{D}$, we know that $b_0 = e - v_0$, where v_0 is the number of white vertices, and that $b_1 = e - v_1$, where v_1 is the number of black vertices. Moreover, $2\sum_{f_{\mathcal{D}}(P)=\infty}(m_P(f_{\mathcal{D}})) = 2e$, so $b_\infty = e - f$. We have then that:

$$\chi(X) = -(-2e + b_0 + b_1 + b_\infty) = -(-2e + e - v_0 + e - v_1 + e - f) = v - e + f$$

□

3.2 The monodromy of dessins

3.2.1 Covering Spaces

Another characterization of dessins can be seen using the classification of covering spaces in algebraic topology. Lets start with some notions of the theory of covering spaces. Note that in this section, the covering maps are not branched.

Proposition 3.12. *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering space and consider a map $f : (Y, y_0) \to (X, x_0)$, with Y path-connected and locally path-connected. A lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ exists if and only if*

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$$

Moreover, if this lift exists, it is unique.

Proof. See Propositions 1.33 and 1.34 in Hatcher [7]. □

In the previous proposition we can see the deep relation of the covering spaces and the subgroups of $\pi_1(X, x_0)$. This correspondence arises from the map that assigns to each covering $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. We would like this map to be surjective, so for every subgroup of $\pi_1(X)$ we would have a covering. For this to occur, we have to ask some mild conditions to the space X, namely , we need X to be semilocally simply connected. This means that for each point $x \in X$, there is a neighborhood U such that the map induced by the inclusion $\pi_1(U, x) \to \pi_1(X, x)$ is trivial. In these conditions, we have the following:

Proposition 3.13. *Suppose X is a path-connected, locally path-connected and semilocally simply-connected space. Then for every subgroup $H \in \pi_1(X, x_0)$ there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, \widetilde{x}_0)) = H$.*

Proof. See Proposition 1.36 in Hatcher [7]. □

Remark 3.14. Note that if X is a connected manifold and, in particular, if X is a connected Riemann surface, then it is always path connected, locally path-connected and semilocally simply-connected.

We have then that two coverings $p_1 : (X_1, x_1) \to (X, x)$ and $p_2 : (X_2, x_2) \to (X, x)$ are isomorphic (as coverings) if the associated subgroups H_1 and H_2 are isomorphic. Now we would like to eliminate the base point. With some effort (which can be seen in Miranda [3] or Munkres [8]), we can see that two coverings (without base point) are isomorphic if the associated subgroups are conjugated.

Consider the fiber $p^{-1}(x_0)$ and denote by $\{x_1, ..., x_d\}$ the d points on it. Every loop γ on X based on x_0 can be lifted to d loops $\gamma_1, ..., \gamma_d$, where $\gamma_i(0) = x_i$ for every $i = 1, ..., d$.

Consider the points $\gamma_i(1)$ which form the entire fiber $p^{-1}(x_0)$, so $\gamma_i(1) = x_j$ for some j. This defines a permutation σ which send i to $\sigma(i) = j$.

Definition 3.15. This permutation $\sigma \in \Sigma_d$ only depends on the homotopy class of the loop γ, and so, we have a group homomorpshim

$$\rho : \pi_1(X, x_0) \to \Sigma_d$$

We call this homomorphism the **monodromy representation** of the covering.

We call **monodromy action** of the covering to the action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$.

Remark 3.16. Let $p : \widetilde{X} \to X$ be a d-sheeted covering space, with associated subgroup H, and let $x_0 \in X$.Then, the stabilizer subgroup of the monodromy action corresponding to x_0 is exactly H, and thus, the fiber $p^{-1}(x_0)$ is isomorphic to the space of cosets $H \setminus \pi_1(X, x_0)$, and the index $[\pi_1(X, x_0) : H]$ is d.

3.2.2 The cartographic group of a dessin.

Proposition 3.17. *Let $\rho : \pi_1(X, x_0) \to \Sigma_d$ be the monodromy representation of a covering map $p : \widetilde{X} \to X$ of degree d, where $\widetilde{X}$ is connected. The image of ρ is a subgroup of Σ_d that acts transitively on the fiber $p^{-1}(x_0)$.*

Proof. Consider two indices i, j and the two points x_i, x_j. Since $\widetilde{X}$ is path-connected, there is a path $\tilde{\gamma}$ which joins x_i to x_j. Consider the image of $\tilde{\gamma}$ by p, $\gamma = p \circ \tilde{\gamma}$, which is a loop in X based at x_0. With the above construction, is clear that the homotopy class of γ generates a permutation in Σ_d that sends i to j. □

Consider now a Belyi pair (S, f). The map $f : S \to \mathbb{P}^1$ is a ramified covering, so its restriction $f : S \setminus f^{-1}(\{0, 1, \infty\}) \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is a covering space of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ in the conditions of proposition 3.13. The fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, 1/2) = F_2$, the free group with two generators, as in the following picture, where the generators are α and β.:

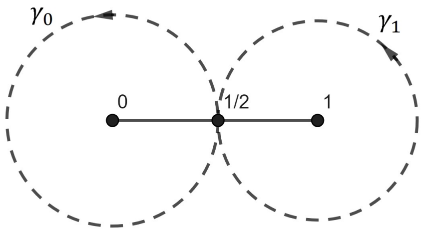


FIGURE 3.4: Two generators.

Proposition 3.18. *Dessins of degree d are in one-to-one correspondence with conjugacy classes of subgroups of F_2 with finite index d, whose image acts transitively on a set of d elements .*

Remark 3.19. Moreover, the category of dessins d'enfants is equivalent to the category of transitive (right) actions of F_2 on finite sets, whose maps are maps $f : Y \to Y'$ such that for all $\varphi \in F_2$ and for all $y \in Y$, $\varphi(f(y)) = f(\varphi(y))$.

proof of the proposition 3.18. Consider a dessin (S, f) . As we have seen, f restricted to $S \setminus f^{-1}(\{0, 1, \infty\})$ is an unbranched d-sheeted covering

$$f : S \setminus f^{-1}(\{0, 1, \infty\}) \to \mathbb{P}^1 \setminus \{0, 1, \infty\} \cong \mathbb{C} \setminus \{0, 1\}$$

This covering map has associated a subgroup H of $F_2 = \pi_1(\mathbb{C} \setminus \{0, 1\}, 1/2)$ that acts transitively on the fiber $f^{-1}(1/2)$, and such that $[F_2 : H] = d$. Moreover, by Proposition

3.13 every subgroup in these conditions defines a covering map. Since the monodromy action is transitive, the preimage of the segment $[1, 2]$ is connected and so, every subgroup in these conditions defines a dessin.

Furthermore, consider a dessin (S, f) and the transitive action of F_2 on the fiber $Y = f^{-1}(x_0)$, and consider a morphism of dessins $\Phi : (S, f) \to (S', f')$. We can define a base point in S' by taking $\Phi(x_0)$. The morphism Φ induces an inclusion $H \to H'$ between the two subgroups associated to the coverings f and f' respectively. Since $Y \cong F_2/H$, and H is contained in H', we have that H' is the union of some cosets of H, and thus, the cosets of H' form a partition of F_2/H. On Y, the relation is

$$a \sim b \Leftrightarrow \exists h' \in H', , a^{h'} = b$$

This quotient is F_2/H', and so, F_2 acts on $Y' \cong F_2/H'$ by the induced action. This means that the morphism Φ induces a morphism of actions, which is simply a map

$$\Phi' : Y \to Y'$$

that commutes with the action. Since the action is transitive, this map must be surjective. □

Definition 3.20. A dessin of degree d can be seen as a morphism

$$M_f : F_2 \to \Sigma_d$$

We call the image $M_f(F_2) \subset \Sigma_d$ of F_2 by this homomomorphism the **cartographic group** of the dessin. In the next section we will discuss a constructive way to find the cartographic group of a dessin.

Let (S, f) be a dessin d'enfants of degree N. Its monodromy representation, and so, the cartographic group of the dessin, is determined by two permutations σ_0 and σ_1, the images by M_f of the two generators of the free group. Lets describe these permutations:

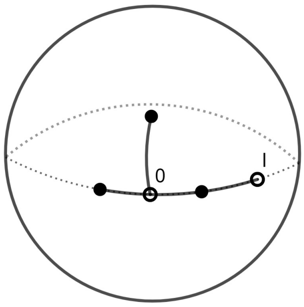
Consider the dessin $(S_f, \mathcal{D}_f)$, which has N edges, and label them with integers from 1 to N. Draw small circles around the white vertices and consider the following permutation σ_0:
$\sigma_0(i) = j$ if j is the edge that follows i under a clockwise rotation around that white vertex. Similarly, draw small circles around the black vertices and consider the permutation σ_1 which send i to the following edge under a clockwise rotation.

Definition 3.21. We call to the pair (σ_0, σ_1) the **permutation representation** pair of the dessin.

Note that this definition makes sense since every edge has one vertex black and one white and, as $\mathcal{D}$ is connected, the subgroup $< \sigma_0, \sigma_1 > \subset \Sigma_N$ must act transitively in the set of edges. Moreover, if γ_0 and γ_1 are the loops (based in $1/2 \in \mathbb{P}^1$) that generate $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, 1/2)$, the images satisfy that $M_f(\gamma_0) = \sigma_0^{-1}$ and $M_f(\gamma_1) = \sigma_1^{-1}$. This happens because in a neighbourhood of $f^{-1}(0)$ the map f is of the form $z \mapsto z^n$, so if we take $x_j \in f^{-1}(1/2)$ such that x_j lies in the edge j of $\mathcal{D}_f$, the lift of γ_0 ends at $x_{\sigma_0^{-1}(j)}$. A similar argument proves that $M_f(\gamma_1) = \sigma_1^{-1}$. So we have the following:

Proposition 3.22. *The cartographic group of a dessin d'enfants (S, f) and the permutation representation pair of $(S_f, \mathcal{D}_f)$ are determined by each other.*

Example 3.2 (Shabat polynomials)**.** *In general, a Shabat polynomial is a polynomial whose branch values are only* 0 *and* 1 *(and ∞). The dessins associated to this kind of morphisms are always trees over the Riemann Sphere. Consider a dessin given by a tree drawn on the Riemann Sphere, for example*



The associated Belyi map must be a polynomial since the dessin has only one face and so, the map must have only one pole in ∞. Since the degree of the vertex 0 *is 3, and the degree of the vertex* 1 *is 1, the polynomial must be of the form*

$$f(z) = Cz^3(z-1)$$

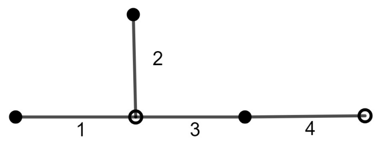
Since one of the black vertices has degree 2*, we know that f must have one and only one point α of multiplicity* 2*. If we find α and we make $f(\alpha) = 1$, we will obtain the constant C. In order to find α, lets compute the derivative $f'(z)$.*

$$f'(z) = C(4z^3 - 3z^2) = Cz^2(4z - 3)$$

Thus, the point α is $\alpha = \frac{3}{4}$. Making $f(\frac{3}{4} = 1)$ we obtain that

$$f_{\mathcal{D}}(z) = \frac{-256}{27} z^3(z-1)$$

Now, labeling the edges of the graph we can see that the cartographic group of this dessin



is given by

$$\sigma_0 = (1, 2, 3)$$
$$\sigma_1 = (3, 4)$$

3.3 Regular Dessins

In this section we talk about the most symmetric dessins, the regular dessin d'enfants.

Definition 3.23. An automorphism of a dessin is an isomorphism $\varphi : (X, \mathcal{D}) \to (X, \mathcal{D})$.

As we have seen, dessins can be seen either as graphs embedded in surfaces, as Belyi maps, as field extensions or as transitive F_2-actions on finite sets. In all these contexts the notion of automorphism makes sense:

1. An automorphism of a Belyi map (S, f) is a map $\varphi : S \to S$ such that $f \circ \varphi = f$.

2. An automorphism of a field extension $\mathbb{C}(S)$ of $\mathbb{C}(z)$ is a map $\varphi : \mathbb{C}(S) \to \mathbb{C}(S)$ such that $\varphi|_{\mathbb{C}(z)} = Id$.

3. An automorphism of an F_2-action is a map commuting with it.

The set of all automorphisms of a dessin $(X, \mathcal{D})$ has the structure of a group, which is denoted by $\mathrm{Aut}(\mathcal{D})$.

Theorem 3.24. *Let $(X, \mathcal{D})$ be a dessin, (S, f) be the associated Belyi pair and $H \subset F_2$ the subgroup associated with the monodromy action. Then, the following are equivalent:*

(1) The order of $\mathrm{Aut}(\mathcal{D})$ *equals the degree of* $(X, \mathcal{D})$.

(2) The covering $f : S \to \mathbb{P}^1$ is a normal (or Galois) covering of the sphere, that is, there exists a subgroup G of $\mathrm{Aut}(\mathcal{D})$ *such that f is equivalent to $S \to S/G$.*

(3) The extension $\mathbb{C}(S)/\mathbb{C}(z)$ is a Galois extension.

(4) The group H is a normal subgroup of F_2, and the automorphism group $\mathrm{Aut}(\mathcal{D})$ is isomorphic to $N_{F_2}H/H$.

(5) The monodromy action is free, that is, if an element $g \in G$ fixes a point, then g is the identity .

(6) The order of the cartographic group is equal to the degree of the dessin.

(7) The automorphism group $\mathrm{Aut}(\mathcal{D})$ acts transitively on the set of edges of $\mathcal{D}$.

In these conditions, the dessin $(X, \mathcal{D})$ is said to be a ***Regular dessin****.*

Proof. $(1 \Leftrightarrow 7)$ is obvious, since the degree of $\mathcal{D}$ is the number of edges of $\mathcal{D}$.

$(1 \Leftrightarrow 2)$ Let G be the automorphism group of the cover. Since its elements are determined by the image of an unramified point, the number of elements of G is at most the degree of the dessin. A similar proof of the one which can be seen in Proposition 2.21 of Girondo and González-Diez [5] proves that the quotient by G is an holomorphic map to another Riemann surface, and so, f must factor through this quotient map, giving the diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}^1 \\ S/G \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

An automorphism that fixes an unramified point of f must fix them all, and so, G acts properly on the unramified points of f. Therefore, the degree of π must be $|G|$. Since we are asuming (1), the degree of f is also $|G|$ and so, the degree of $\bar{f}$ is 1, that is, $\bar{f}$ is an isomorphism, and π is equivalent to f.

Conversely, if the map $S \to S/G$ is equivalent to f, then G is the group of automorphisms of the cover. Therefore, its order is, at least, the order of the fibers, which is the degree of the dessin. Since the order of the automorphism group is always smaller that de degree of the dessin, we are done.

($2 \Leftrightarrow 3$) Suppose that f is equivalent to $\pi : S \to S/G$ being the map π the one seen in the previous part of the proof. Since $\pi^* : \mathbb{C}(S/G) \to \mathbb{C}(S)$ is a field homomorphism, $\mathbb{C}(S/G)$ must be a subfield of $\mathbb{C}(S)$. Consider $G^* = \{\varphi^* : \varphi \in G\}$, which is a subgroup of the Galois group $\mathrm{Gal}(\mathbb{C}(S)/\mathbb{C}(z)$. Since the functions defined on S/G are the ones that G preserves, the fixed field of G^* must be $\mathbb{C}(S/G)$. Since π is the Belyi map, π^* is the inclusion of $\mathbb{C}(\mathbb{P}^1) \cong \mathbb{C}(z)$ on $\mathbb{C}(S)$ and $\mathbb{C}(S/G) = \mathbb{C}(z)$. Therefore, G^* is the whole Galois group, and the extension is Galois.
Now, suppose the extension $\mathbb{C}(S)/\mathbb{C}(z)$ is Galois. If G^* is the Galois group of the extension, the fixed field of G^* must be $\mathbb{C}(z)$. Every automorphism in G^* corresponds to an automorphism of S that preserves z, that is, an automorphism of S, f. Therefore, if we call G to the subgroup of $\mathrm{Aut}(S)$ corresponding to this Galois group, we have that $\mathbb{C}(z) = \mathbb{C}(S/G)$, and so, the Belyi map f is equivalent to $S \to S/G$.

($1 \Rightarrow 4$) Suppose we have a transitive F_2-action on a set X, and an automorphism $\varphi : X \to X$. Let $x_0 \in X$, and let H be its stabilizer. We claim that φ is determined by $\varphi(x_0)$, indeed, since the action is transitive, every $x \in X$ is x_0^g for some $g \in F_2$. Then,

$$\varphi(x) = \varphi(x_0^g) = \varphi(x_0)^g$$

Now, let $h \in F_2$ be such that $\varphi(x_0) = x_0^h$. Then, $\varphi(x_0^g) = \varphi(x_0)^g = x_0^{hg}$, for every $g \in F_2$. In particular, if g_1, g_2 are such that $x_0^{g_1} = x_0^{g_2}$,

$$x_0^{hg_1} = \varphi(x_0^{g_1}) = \varphi(x_0^{g_2}) = x_0^{hg_2}$$

if and only if

$$x_0^{hg_2g_1^{-1}h^{-1}} = x_0$$

which happens if and only if $hg_2g_1^{-1}h^{-1} \in H$. This is equivalent to

$$hHh^{-1} = H$$

In other words, $h \in N_{F_2}(H)$. Since h induces the identity if and only if $x_0^h = x_0$, the automorphism group is isomorphic to $N_{F_2}(H)/H$. We are assuming that $\mathrm{Aut}(\mathcal{D})$ has order equal to the degree of the dessin, which is the index of H, and so, $N_{F_2}(H) = F_2$ and H is a normal subgroup of F_2.

(4 ⇒ 5) Let H be, as before, the stabilizer of x_0. Then the stabilizer of x_0^g is $g^{-1}Hg$. Since H is a normal subgroup of F_2, the intersection of all stabilizers is H. Then, the morphism

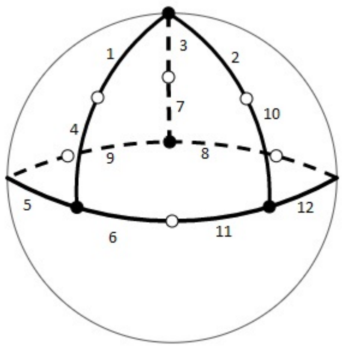
$$\varphi : F_2 \to \Sigma_d$$

that has the cartographic group as image, has H as its kernel. Therefore, a permutation stabilizes a point if and only if it is in the image of H^g for some g, and so, it is in the kernel of the action.

(5 ⇒ 6) The order of the cartographic group is the product of the cardinality of the orbit by the order of the stabilizer of a point. The order of the orbit is the degree of the dessin, since the action is transitive, and by hypothesis, the order of the stabilizer is 1.

(6 ⇒ 1) This is a particular case of the proposition 2.66 in Girondo and González-Diez [5]. □

Example 3.3. *Consider the dessin drawn in figure 3.2(a). The edges can be labeled as in the following picture:*



It is well known that the symmetry group of orientation preserving automorphisms of thetrahedron has order 12 *and, thus, the number of automorphisms of this dessin is* 12*, and so, the dessin is Regular. Anyway, one can compute the cartographic group considering its generators:*

$$\sigma_0 = (1,4)(2,10)(3,7)(5,9)(6,10)(8,12)$$
$$\sigma_1 = (1,2,3)(4,5,6)(7,8,9)(10,11,12)$$

A computer will immediately compute the subgroup of Σ_12 *generated by* σ_0 *and* σ_1 *which, obviously, has* 12 *elements.*

Corollary 3.25. *A regular Dessin can not have leaves on its graph.*

3.4 The action of $\mathrm{Gal}(\overline{\mathbb{Q}})$

We have already seen how the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}})$ acts on compact Riemann surfaces and their morphisms. Since Belyi pairs (S, f) are defined over $\overline{\mathbb{Q}}$, $\mathrm{Gal}(\overline{\mathbb{Q}})$ acts on them in the same way: by conjugation of the polynomials describing these objects.

Theorem 3.26. *Let $(X, \mathcal{D})$ be a dessin. The following properties are invariant under the action of the absolute Galois group:*

(1) The number of edges, and so, the degree of the dessin.

(2) The number of white vertices, black vertices and faces.

(3) The degree of the white vertices, the black vertices and the faces.

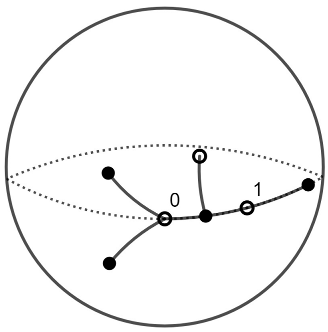
(4) The genus.

(5) The cartographic group.

(6) The automorphism group.

Proof. This is just a consequence of the theorem 2.16. □

Example 3.4. *Consider the following dessin. Assume the white vertex of degree 3 is*



placed at 0, the white vertex of degree 2 is placed at 1 and the third white vertex is placed at $z = a$. Then, the corresponding Shabat polynomial is of the form

$$f(z) = Cz^3(z-1)^2(z-a)$$

In the same way we did in the example 3.2, we compute the derivative of f

$$f'(z) = C(z^2(z-1)(6z^2 + (-5a-4)z + 3a))$$

Since the graph has a black vertex of degree 3*, we now that there must be a branch point* α *of order* 3*. This must occur as a double root of* f' *and so, the discriminant of*

$$P(z) = 6z^2 + (-5a-4)z + 3a)$$

must vanish, that is

$$25a^2 - 32a + 16 = 0$$

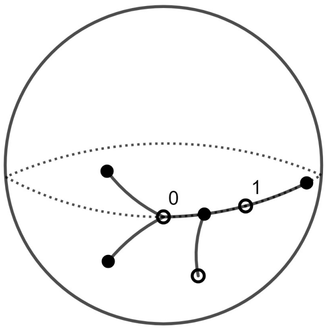
hence $a = a_1 = \frac{4}{25}(4+3i)$ *or* $a = a_2 = \frac{4}{25}(4-3i)$*. These* a_1 *and* a_2 *determine two Belyi functions* f_1 *and* f_2*. Their corresponding constant* C_j *can be deduced from the condition* $f_j(\alpha_j) = 1$*, where* $\alpha_j = \frac{5a_j+4}{12} = \frac{3\pm i}{5}$ *are the roots of* $P(z)$*. Then, the corresponding Belyi functions are*

$$f_1(z) = \frac{3+i}{5}z^3(z-1)^2(z-\frac{4}{25}(4+3i))$$

and

$$f_1(z) = \frac{3-i}{5}z^3(z-1)^2(z-\frac{4}{25}(4-3i))$$

These two Belyi functions, could determine the same dessin, but the dessin $\bar{\mathcal{D}}$ *of the following picture has also six edges, and the same collection of vertex-degrees as* $\mathcal{D}$*, so*



its Belyi function is either f_1 *or* f_2*.*

There is no orientation-preserving automorphism of the sphere sending $\mathcal{D}$ *to* $\bar{\mathcal{D}}$ *so* f_1 *and* f_2 *correspond to two non-equivalent dessins. Since* $\bar{\mathcal{D}} = \mathcal{D}^\sigma$ *for any* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}})$ *that sends* i *to* $-i$*, the two dessins must lie in the same Galois orbit. In fact, since the* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}})$ *that send* i *to* $-i$ *are the only elements of the absolute Galois group that act non-trivially on* f_1*, we can deduce that* $\{\mathcal{D}, \bar{\mathcal{D}}\}$ *is a complete Galois orbit.*

Definition 3.27. An action $G \times X \to X$ is said to be faithful if the only element $g \in G$ that fixes all points $x \in X$ is the identity $e \in G$.

It is an important feature of the Galois action on dessins the fact that, for every g, the action is faithful on dessins of genus g.

Theorem 3.28. *The restriction of the action of* $\mathrm{Gal}(\overline{\mathbb{Q}})$ *to dessins of genus* g *is faithful for every* g*.*

Proof. First, let $\mathcal{D}$ be a tree on the Riemann Sphere, that is f is a Shabat polynomial. Let $\alpha \in \overline{\mathbb{Q}}$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}})$ such that $\sigma(\alpha) \neq \alpha$, and consider a polynomial p_α ramified exactly at $\{0, 1, \infty\}$ and such that the ramification numbers $m_0(p_\alpha)$, $m_1(p_\alpha)$ and $m_\alpha(p_\alpha)$ are pairwise distinct. In these conditions, we can find a polynomial $q = q_\alpha \in \mathbb{Q}[x]$ such that $P_\alpha = q \circ p_\alpha$ is a Shabat polynomial.

The automorphism σ sends P_α to $P_\alpha^\sigma = q^\sigma \circ p_\alpha^\sigma = q \circ p_{\sigma(\alpha)}$. If the two dessins $(\mathbb{P}^1, P_\alpha)$ and $(\mathbb{P}^1, P_\alpha^\sigma)$ were equivalent, there would exist an automorphism T such that the diagram

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[r, "T"] \arrow[dr, "P_\alpha^\sigma"'] & \mathbb{P}^1 \arrow[d, "P_\alpha"] \\
 & \mathbb{P}^1
\end{tikzcd}$$

commutes. Since $T(\infty) = \infty$, T must be of the form $T(z) = az + b$, and so,

$$q(p_\alpha(az + b)) = P_\alpha(az + b) = P_\alpha^\sigma(z) = q(p_{\sigma(\alpha)}(z))$$

From the lemma 4.50 in Girondo and González-Diez [5], we can deduce that there exist constants c, d such that

$$p_\alpha(az + b) = cp_{\sigma(\alpha)}(z) + d$$

This equation shows that the ramification points of P agree with those of $p_{\sigma(\alpha)}$, and that they are the image by $T^{-1}(z) = \frac{z-b}{a}$ of those of p_α. It follows that $b = 0$, $a = 1$ and $\sigma(\alpha) = \alpha$, which is a contradiction. This shows that the Galois action is faithfull on Shabat polynomials, and so, on dessins of genus 0.

Now, the faithfullness of the action on dessins of genus 1 follows easily from the fact that the moduli space of this Riemann surfaces is isomorphic to $\mathbb{C}$: If C_λ is the Riemann surface corresponding to the curve $y^2 = x(x-1)(x-\lambda)$, the injective j-invariant $j : \mathbb{C} \setminus \{0, 1\} \to \mathbb{C}$ defined by

$$j(\lambda) = \frac{1 - \lambda + \lambda^2}{\lambda^2(\lambda - 1)^2}$$

classifies the Riemann surfaces of genus 1.

Let $\sigma \in \mathrm{Gal}(\mathbb{C})$ and $z \in \mathbb{C}$ such that $\sigma(z) \neq z$. Take λ with $j(\lambda) = z$. Clearly, $C_\lambda^\sigma = C_{\sigma(\lambda)}$ has j-invariant

$$j(\lambda^\sigma) = j(\lambda)^\sigma = \sigma(z) \neq z$$

and so, C_λ can not be isomorphic to C_λ^σ.

The proof for $g > 1$ can be seen in Girondo and González-Diez [5]. $\square$

There are some other invariants of the Galois action on dessin d'enfants that will not be discussed here. One would like to have enough invariants to separate all orbits of the Galois action, but it is an open problem to construct this complete collection.

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